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РЕЗЮМЕ

ОЧЕРЕДИ ИЗ ГРУПП

Ф. Г. Фостер (Лондон)

Пусть прибывают в очередь группы из r единиц, которые обслуживаются по одному. Пусть промежутки времени между прибытием групп имеют любое данное распределение вероятностей, и времени обслуживания имеют показательное распределение. В работе определено предельное распределение числа ожидающих единиц во время прибытия новой группы и времени ожидания первого члена группы. Рассматриваются также некоторые частные случаи общей проблемы и связь рассмотренной проблемы с обычной теорией очереди, где единицы прибывают по одному.

О ПЕРИОДИЧЕСКИХ РЕШЕНИЯХ НЕКОТОРЫХ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА С ПЕРИОДИЧЕСКИМИ КОЭФФИЦИЕНТАМИ

И. Бихари (Будапешт)

Доказательство существования периодических решений уравнений с периодическими коэффициентами встречается с серьезными трудностями уже и в случае линейных уравнений. В случае нелинейных уравнений трудности лишь значительно возрастают. Для этой цели служат сложные методы, разработанные ввиду теоретической и практической важности вопроса. Поэтому кажется неожиданным, что в случае уравнения $y''+\varphi(x)f(y)h(y')=0$ (которое не менее специально чем изучавшиеся в литературе) при T-периодичном $\varphi(x)$ какими простыми средствами можно придти к результату. Настоящая работа дает условие существования T- и $2T$-периодичного решения, состоящего из $2n$ четвертьволн, т. е. получаем возможность ознакомиться и с более тонкой структурой периодических решений. Аналогичные результаты получаются для линейного уравнения $y''+[\alpha+\beta\varphi(x)]y=0$ и уравнения $y''+\varphi(x)f(y,y')=0$, где $f(u,v)$ однородная функция первой степени и $\operatorname{sg} f(u,v)=\operatorname{sg} u$. Работа объясняет, почему линейное уравнение $y''+\varphi(x)y=0$ в общем случае не имеет периодических решений.

СТАНДАРТНЫЕ ИДЕАЛЫ В СТРУКТУРАХ

Г. Гретцер и Э. Т. Шмидт (Будапешт)

Цель работы обобщением нейтрального идеала определить такой тип идеалов, который играл бы в теории структур роль, подобную роли нормальных делителей в теории групп. При этом требуется, чтобы этот тип идеалов обладал важнейшими свойствами нейтральных идеалов, что делает возможным обобщение теорем о нейтральных идеалах.

Элемент s (идеал S) структуры L называется стандартным, если для любых $x, y \in L$ (для любых идеалов X, Y структуры L) выполняется соотношение

$$x \cap (s \cup y) = (x \cap s) \cup (x \cap y) \quad ((X \cap (S \cup Y) = (X \cap S) \cup (X \cap Y)).$$

Теорема 1. Для элемента s структуры L следующие условия эквивалентны:

(α) элемент s — стандартный;

(β) для всех элементов u и t структуры L, для которых $u \leqq s \cup t$, имеет место соотношение

$$u = (u \cap s) \cup (u \cap t);$$

(γ) отношением конгруентности L является отношение Θ_s, для которого $x \equiv y\,(\Theta_s)$ выполняется в том и только в том случае, если $(x \cap y) \cup s_1 = x \cup y$ для некоторого $s_1 \leqq s$;

(δ) для любых элементов x и y структуры L

(i) $$s \cup (x \cap y) = (s \cup x) \cap (s \cup y),$$

(ii) из соотношений $s \cup x = s \cup y$ и $s \cap x = s \cap y$ следует равенство $x = y$.

Аналогичная теорема может быть получена для стандартных идеалов. Из этих двух теорем могут быть получены наиболее важные свойства стандартных элементов и идеалов (теоремы 3—6, леммы 1—9).

Доказывается, что понятие стандартного и нейтрального элемента совпадает в случае слабомодулярных структур, являющихся общим обобщением модулярных структур и структур с относительным дополнением (теоремы 7 и 8).

Авторы доказывают, что в структурах с отрезочным дополнением (где всякий отрезок $[0, a]$, как структура, дополнительный) ядро гомоморфизма и понятие стандартного идеала совпадают (теорема 11), что является обобщением одной теоремы Биркгофа [6]. Отсюда получается обобщение одного результата Дилуэрта [8] и одной теоремы Ванга [34].

Приводится ряд примеров того, как с помощью „словаря"

подгруппа → идеал
нормальный делитель → стандартный идеал
фактор-группа → фактор-структура
групповое действие → объединение

ряд теорем теории групп может быть переформулирован для структур. Так получаются теоремы изоморфизмы, лемма Цассенхауса, теорема Жордана—Гёльдера—Шрейера. Перефразировка проблемы Шрейера о расширении группы также приводит к разрешимой проблеме.

Из дальнейших результатов отметим две теоремы (теорема 21 и 23) о совпадении идеалов, удовлетворяющих первой теореме об изоморфизме, и нейтральных идеалов в специальном классе структур с отрезочным дополнением и в модулярных структурах с локально конечной длиной и нулевым элементом.

ОБ ОДНОМ СВОЙСТВЕ „СЕМЕЙСТВ" МНОЖЕСТВ

П. Эрдёш и А. Хайнал (Будапешт)

Система множеств $\mathfrak{F}$ называется системой со свойством **B**, если существует такое множество B, для которого $B \cap F = 0$ и $F \subseteq\!\!\!|\; B$ для всякого $F \in \mathfrak{F}$. $\mathfrak{F}$ называется системой со свойством $\mathbf{B}(s)$, если существвет множество B, для которого $0 < \overline{\overline{B \cap F}} < s$, где s любая мощность. $\mathfrak{F}$ обладает свойством $\mathbf{C}(q, r)$, если для всякого $\mathfrak{F}' \subseteq \mathfrak{F}$ из $\overline{\overline{\mathfrak{F}'}} \geqq q$ следует $\overline{\overline{\bigcap_{F \in \mathfrak{F}'} F}} < r$.

Если существует мощность p, для которой в случае $F \in \mathfrak{F}$ $\overline{\overline{F}} = p$, то это обозначается так: $p(\mathfrak{F}) = p$.

Вводятся следующие символы;

$\mathbf{M}(m, p, q, r) \rightarrow \mathbf{B}$ обозначает тот факт, что каждая система множеств $\mathfrak{F}$, для которой $\overline{\overline{\mathfrak{F}}} = m$, $p(\mathfrak{F}) = p$, и которая обладает свойством $\mathbf{C}(q, r)$, должна обладать свойством **B**;

$\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ обозначает тот факт, что всякая система множеств, для которой $\overline{\overline{\mathfrak{F}}} = m$, $p(\mathfrak{F}) = p$ и которая обладает свойством $\mathbf{C}(2, r)$, должна обладать свойством $\mathbf{B}(s)$;

$\mathbf{M}(m, p, q, r) \not\rightarrow \mathbf{B}$, $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ обозначает отрицание соответствующих фактов.

Исходя из частных результатов Миллера, доказанных в [1], с помощью общей гипотезы континуума дается почти полная дискуссия выше упомянутых символов.

Отмечается, что с помощью полученных результатов можно сделать несколько выводов относительно возможных теоретико-множественных обобщений теоремы Тихонова, в частности, также с помощью гипотезы континуума доказывается, что топологическое произведение $\aleph_k$ 1-компактных дискретных топологических пространств не k-компактно, где k любое целое число (см. теорему 11).

Кроме того формулируется ряд нерешенных проблем.

О НЕКОТОРЫХ ЗАМЕЧАНИЯХ И ПРОБЛЕМАХ В СВЯЗИ С ОКРАШИВАНИЕМ ГРАФОВ

Ян Мыцельский (Вроцлав)

Первая часть работы рассматривает импликации девяти утверждений. Среди них наряду с утверждениями относительно топологического произведения бикомпактных пространств Хаусдорфа фигурирует и следующее: если каждый частичный граф некоторого графа может быть окрашен n цветами, то это же имеет место для полного графа.

Вторая часть работы доказывает эквиваленцию четырех утверждений. Эквиваленцию первых двух выражений высказывает теорема Куратовского, занимающаяся вложимостью в плоскость конечных графов.

ТЕОРЕМЫ О МАКСИМУМЕ И МИНИМУМЕ И ОБОБЩЕННЫЕ ФАКТОРЫ ГРАФОВ

Т. Галлаи (Будапешт)

Пусть каждой точке X конечного графа Γ без направления соответствуют неотрицательные целые числа $\varkappa(X)$ и $\varkappa'(X)$. Система дуг, состоящая из дуг, попарно не содержащих общих граней, называется совместимой (относительно $\varkappa$ и $\varkappa'$), если в любую точку X попадает не более $\varkappa(X)$ граничных точек и не более $\varkappa'(X)$ внутренних точек, относящихся к системе дуг. (Под дугой понимается путь или петля. Петля — такая окружность, одна из точек которой является двойной граничной точкой.) $\nu_{\max}$ максимум числа дуг совместимых систем дуг. $q = q(A, B, C)$ обозначает следующую систему весов: A, B и C любые такие множества точек графа Γ, что любая точка Γ встречается в одном и только одном из них. Точкам A, B и C соответствуют веса 0, 1 и 1/2. Граням, обе граничные точки которых относятся к A, сопоставляется вес 1; граням, одна из граничных точек которых принадлежит A, а другая C, сопоставляется вес 1/2. Система весов q обладает тем свойством, что сумма весов, относящихся к любой грани Γ и ее граничным точкам, $\geqq 1$. Всем таким системам весов q подходящим образом сопоставляется значение $S(q)$, зависящее от $\varkappa$ и $\varkappa'$. $S_{\min}$ минимум этих значений $S(q)$. Утверждение основной теоремы: $\nu_{\max} = S_{\min}$. Из основной теоремы выводятся теоремы, аналогичные „теореме n цепей" Менгера. Из основной теоремы получается также ряд теорем относительно существование обобщенных факторов. Под обобщенным фактором или $(\varkappa, \varkappa')$-фактором понимается такая совместимая система дуг, из граничных точек относящихся к которой дуг в любую точку X попадает точно $\varkappa(X)$ граничных точек. Эти теоремы в качестве специального случая содержат ряд известных теорем относительно обычных факторов.

ОБ ОБОБЩЕНИИ ТЕОРЕМЫ ПУЛЕН—ЭРМИТА, ОТНОСЯЩЕЙСЯ К ВЕЩЕСТВЕННЫМ КОРНЯМ МНОГОЧЛЕНОВ С ВЕЩЕСТВЕННЫМИ КОЭФФИЦИЕНТАМИ

Н. Обрешков (София)

Автор, обобщая хорошо известную теорему Пулен—Эрмита, доказывает, что если все корни многочлена

(1) $$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

вещественны и комплексные корни многочлена

$$g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$$

с вещественными коэффициентами расположены в области

(2) $$|\arg z| \leqq \frac{1}{\sqrt{n}};$$

то все корни многочлена

$$b_0 f^{(m)}(x) + b_1 f^{(m-1)}(x) + \cdots + b_m f(x)$$

вещественны.

Из приложений этой теоремы упомянем следующее:

Если коэффициенты многочлена (1) вещественны и его корни расположены в области (2), то все корни многочлена

$$\frac{a_0}{n!}x^n + \frac{a_1}{(n-1)!}x^{n-1} + \cdots + \frac{a_{n-1}}{1!}x + a_n$$

вещественны.

О СУММЕ СТЕПЕНЕЙ КОМПЛЕКСНЫХ ЧИСЕЛ

Ф. В. Аткинсон (Торонто)

Цель работы доказать гипотезу П. Турана, согласно которой, если выполняется (1), то независимо от n

$$\text{(I)} \qquad \max_{\nu=1,\ldots,n} \left| z_1^\nu + \cdots + z_n^\nu \right| > \frac{1}{6}.$$

Наилучшая постоянная в (I) не известна. Туран в книге[1] доказал лишь более слабое неравенство, когда вместо $\frac{1}{6}$ стоит $\log 2\left(\sum_{\nu=1}^{n}\frac{1}{\nu}\right)^{-1}$, которое позднее де Брёйн и Утияма улучшили до $(1-\varepsilon)\frac{\log\log n}{\log n}$, если $n > n_0(\varepsilon)$. В книге Турана было показано, что уже применяя его результат, можно дать очень простой метод приближенного решения алгебраических уравнений, применение (I) приводит к существенному упрощению метода.

ОБ ОБОБЩЕНИИ ОДНОГО НЕРАВЕНСТВА ПОЙА И СЕГЁ

Э. Макаи (Будапешт)

Неравенство, данное формулами (1) и (2) работы, и его обобщение для гильбертого пространства, данное формулой (6), дали В. Грёйб и В. Рейнболдт, опиравшиеся при доказательстве на теорию линейных операторов. Настоящая работа дает элементарное доказательство этих же неравенств.

[1] P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen* (Budapest, 1953).

НЕКОТОРЫЕ ИНТЕРПОЛЯЦИОННЫЕ СВОЙСТВА МНОГОЧЛЕНОВ ЭРМИТА

К. К. Матур и А. Шарма (Лукноу, Индия)

Пусть $x_1, x_2, \ldots, x_n$ обозначают корни многочлена Эрмита $H_n(x)$. Авторы доказывают, что при четном n существует единственный (0, 2)-интерполяционный многочлен $R_n(x)$ не выше $2n-1$-ой степени, удовлетворяющий условиям

$$R_n(x_\nu) = \alpha_\nu, \quad \left\{\frac{d^2}{dx^2} R_n(x)\right\}_{x=x_\nu} = \beta_\nu \qquad (\nu = 1, 2, \ldots, n).$$

Здесь $\alpha_1, \alpha_2, \ldots, \alpha_n$ и $\beta_1, \beta_2, \ldots, \beta_n$ любые числа. При нечетном n такие многочлены, вообще говоря, не могут быть однозначно определены.

Аналогичную теорему авторы доказывают для так называемых (0, 1, 3)-интерполяционных многочленов.

В обоих случаях при четном n авторы приводят явный вид интерполяционных многочленов.

О КРУГОВЫХ И ШАРОВЫХ ОБЛАКАХ

А. Хеппеш (Будапешт)

Л. Фейеш Тот назвал k-слоевым шаровым облаком множество расположенных между двумя параллельными плоскостями не вклиняющихся друг в друга шаров, если каждая прямая, перпендикулярная к параллельным плоскостям, содержит внутреннюю или граничную точку по крайней мере k шаров, иначе говоря, если шары образуют относительно прямых, перпендикулярных к параллельным плоскостям, „k-кратно непроходимую стену". Аналогичным образом может быть определено на плоскости понятие k-слоевого кругового облака. Под толщиной кругового или шарового облака понимается расстояние между параллельными прямыми или плоскостями, между которыми находится облако. Фейеш Тот [1] доказал, что минимум толщины однослоевого шарового облака равен $2+\sqrt{2}$ (минимум толщины однослоевого кругового облака, очевидно, равен 2).

В настоящей работе доказывается, что минимум толщины k-слоевого кругового облака

$$d_k = (k-1)\sqrt{3} + 2,$$

а минимум толщины k-слоевого шарового облака удовлетворяет неравенству

$$D_k \leqq \left(k + \left[\frac{k-1}{2}\right]\right)\sqrt{3} + 2, \quad \text{если} \quad k \geqq 2.$$

ОБ АБСОЛЮТНОЙ СХОДИМОСТИ ТРИГОНОМЕТРИЧЕСКИХ РЯДОВ С ПРОБЕЛАМИ

П. Сюс (Будапешт)

В работе доказываются следующие теоремы:

Теорема 1. Пусть постоянная $K \geqq 2$. Тогда существует такая последовательность натуральных чисел

$$n_1, n_2, \dots \qquad \left(\frac{n_{k+1}}{n_k} \geqq K\right),$$

что для любой монотонно убывающей последовательности $a_1, a_2, \dots$

$$\sum_{k=1}^{\infty} a_k |\sin \pi n_k x| < \infty \tag{1}$$

может выполняться для нецелого x лишь в том случае, если

$$\sum_{k=1}^{\infty} a_k < \infty .$$

Теорема 2. Пусть $n_1, n_2, \dots$ есть последовательность целых чисел, для которой

$$\lim_{k\to\infty} \frac{n_{k+1}}{n_k} = \infty .$$

Тогда существует такая монотонно убывающая последовательность $a_1, a_2, \dots$, что, хотя $\sum_{k=1}^{\infty} a_k = \infty$, ряд (1) сходится на множестве мощности континуума.

Теорема 1 является распространением на некоторые ряды с пробелами одной классической теоремы Фату; теорема 2 утверждает, что такое распространение невозможно, если пробелы последовательности $\{n_k\}$ „слишком большие".

ОБ ОДНОЙ ЭКСТРЕМАЛЬНОЙ ЗАДАЧЕ ТЕОРИИ ИНТЕРПОЛИРОВАНИЯ

П. Эрдёш и П. Туран (Будапешт)

Если

$$1 \geqq x_1 > \dots > x_n \geqq -1, \quad \omega(x) = \prod_{\nu=1}^{n} (x - x_\nu), \tag{1}$$

то фундаментальные многочлены относящегося к (1) интерполирования Лагранжа, как хорошо известно, имеют вид

$$l_k(x) = \frac{\omega(x)}{\omega'(x)(x - x_k)} \qquad (k = 1, 2, \dots, n), \tag{2}$$

а так называемые фундаментальные многочлены второго рода интерполирования Эрмита—Фейера суть

$$\mathfrak{h}_k(x) = (x - x_k)\, l_k(x)^2 \qquad (k = 1, 2, \dots, n). \tag{3}$$

Значение фундаментальных многочленов (3) освещено в работе Фейера [2], там же он показал, что при соответствующем выборе узлов (1)

(4) $$\max_{-1\leqq x\leqq 1}\sum_{k=1}^{n}|\mathfrak{h}_k(x)|<\left(\frac{2}{\pi}+\varepsilon\right)\frac{\log n}{n},$$

если только $n>n_0(\varepsilon)$. В настоящей работе авторы доказывают, что при любом выборе узлов (1)

$$\max_{-1\leqq x\leqq 1}\sum_{k=1}^{n}|\mathfrak{h}_k(x)|\geqq\left(\frac{2}{\pi}-c_1\frac{\log\log n}{\log n}\right)\frac{\log n}{n},$$

где c_1 соответственно выбранная положительная постоянная, асимптотически определяя, таким образом, минимум

$$\max_{-1\leqq x\leqq 1}\sum_{k=1}^{n}\mathfrak{h}_k(x).$$

Аналогичным образом авторы доказывают, что при любом выборе узлов (1)

(5) $$\max_{-1\leqq x\leqq 1}\sum_{k=1}^{n}|l_k(x)|\geqq\left(\frac{2}{\pi}-c_1\frac{\log\log n}{\log n}\right)\log n.$$

Несколько менее слабую чем (5) оценку можно найти в работе С. Бернштейна [1]; однако доказательство авторов основывается на совершенно других соображениях

ПРОБЛЕМЫ И РЕЗУЛЬТАТЫ ОТНОСИТЕЛЬНО ИНТЕРПОЛИРОВАНИЯ. II

П. Эрдёш (Будапешт)

Пусть $-1\leqq x_1<x_2<\cdots<x_n\leqq 1$,

$$\omega(x)=\prod_{k=1}^{n}(x-x_k),\quad l_k(x)=\frac{\omega(x)}{\omega'(x)(x-x_k)}.$$

Доказывается, что

(1) $$\max_{-1\leqq x\leqq 1}\sum_{k=1}^{n}|l_k(x)|>\frac{2}{\pi}\log n-c_1.$$

(1) обостряет теоремы Бернштейна и Эрдёша—Турана. Известно, что (1) не может быть улучшено, так как если x_k узлы многочленов Чебышева $T_n(x)$, то

$$\max_{-1\leqq x\leqq 1}\sum_{k=1}^{n}|l_k(x)|<\frac{2}{\pi}\log n+c_2.$$

Неизвестна система точек, для которой

$$\max_{-1\leqq x\leqq 1}\sum_{k=1}^{n}|l_k(x)|$$

принимает свое наименьшее значение.

ОБ ОДНОЙ ПРОБЛЕМЕ БЭРА И ОДНОЙ ПРОБЛЕМЕ УАЙТХЕДА В ТЕОРИИ АБЕЛЕВЫХ ГРУПП

И. Ротмен (Урбана, США)

Автор исследует абелевы группы F, для которых $\operatorname{Ext}(F, T) = 0$ для всех периодических групп T, и те, для которых $\operatorname{Ext}(F, Z) = 0$, где Z группа целых, называя их B-группами и W-группами, соответственно. В счетном случае известно решение обеих проблем: и те и другие группы являются свободными группами.

Автор получает ряд частичных результатов в случае любой мощности. Типичные результаты:

1. Сепарабельная B-группа стройная.

2. Каждая W-группа сепарабельна, стройная и может быть вложена в полную прямую сумму бесконечных циклических групп как сервантная подгруппа.

УПОРЯДОЧЕННЫЕ ПОЛУГРУППЫ

Л. Фукс (Будапешт)

Работа рассматривает вопрос о вложимости в следующие упорядоченные полугруппы:

P: аддитивная полугруппа неотрицательных действительных чисел;

P_1: вещественный отрезок $[0, 1]$, действие $a \cdot b = \min(a + b, 1)$;

P_1^*: отрезок $[0, 1]$ и символ ∞, действие: $a \cdot b = a + b$ или ∞ в зависимости от того, будет ли $a + b \leqq 1$ или > 1.

Теорема 1. Положительно упорядоченная полугруппа тогда и только тогда может быть вложена в P, если она 1. архимедова, 2. не содержит аномальной пары, и 3. не имеет максимального элемента (если она содержит по крайней мере два элемента).

Теорема 3. Архимедова, естественно упорядоченная полугруппа изоморфна по упорядочению одной из подполугрупп от P, P_1 или P_1^*.

Последняя теорема является общим обобщением теорем Гёльдера [4] и Клиффорда [2].

О СИЛЕ СВЯЗАННОСТИ СЛУЧАЙНОГО ГРАФА

П. Эрдёш и А. Реньи (Будапешт)

Пусть $\Gamma_{n, N}$ случайный граф с n вершинами и N ребрами, без петель и без параллельных ребер. $\Gamma_{n, N}$ получается следующим образом: выбираем случайно N из всех возможных $\binom{n}{2}$ ребер, соединяющих n данные вершины $V_1, V_2, \ldots, V_n$; при этом по предположению все возможные $\binom{\binom{n}{2}}{N}$ выборы этих ребер одинакого вероятны.

Пусть для любого (неполного) графа G $c_n(G)$ обозначает минимальное число k, обладающее следующим свойством: можно вычеркнуть из графа G k подходящих вер-

шин (вместе со всеми ребрами, которые инцидентны с этими вершинами) так, что получается несвязный граф. Пусть $c_l(G)$ обозначает минимальное число l, обладающее следующим свойством: можно вычеркнуть из графа G l ребер так, что получается несвязный граф. Наконец, пусть $d(V_k)$ $(k=1, 2, \ldots, n)$ число ребер, исходящих из вершины V_k (валентность вершины V_k) и положим $c(G)=\min\limits_{1\leq k\leq n} d(V_k)$. Каждая из трех величин $c_p(G)$, $c_l(G)$ и $c(G)$ может быть рассмотрена как мера связанности графа G.

В работе доказывается (теорема 2), что если

$$N(n)=\frac{1}{2}n\log n+\frac{r}{2}n\log\log n+\alpha n+o(n), \tag{1}$$

где r данное целое неотрицательное число и α данное вещественное число, то (обозначая через $\mathbf{P}(\cdot)$ вероятность события в скобках) имеем

$$\lim_{n\to+\infty}\mathbf{P}(c_p(\Gamma_{n,\,N(n)})=r)=1-e^{-\frac{e^{-2\alpha}}{r!}}, \tag{2}$$

и что (2) остается верным также, если в нем вместо $c_p(\Gamma_{n,\,N(n)})$ стоит или $c_l(\Gamma_{n,\,N(n)})$ или $c(\Gamma_{n,\,N(n)})$.

Далее доказывается (теорема 3), что число вершин случайного графа $\Gamma_{n,\,N(n)}$ (где $N(n)$ опять определено с помощью (1)), имеющих валентность r, в пределе при $n\to+\infty$ распределено по закону Пуассона с параметром $\lambda=\dfrac{e^{-2\alpha}}{r!}$.

Частные случаи этих теорем, когда $r=0$, были доказаны авторами настоящей статьи уже раньше в их работе [4].

QUEUES WITH BATCH ARRIVALS. I

By
F. G. FOSTER (London)
(*Presented by* A. RÉNYI)

1. Introduction. The following single server queueing system is considered in this paper:

(i) Batches of exactly r units arrive at the sequence of instants, $\tau_1, \tau_2, \ldots, \tau_n, \ldots$, such that the inter-arrival times, $\tau_{n+1}-\tau_n>0$ $(n=1,2,\ldots)$, are identically distributed independent random variables with common distribution function $F(x)$. Put

$$\varphi(s)=\int_0^\infty e^{-sx}dF(x), \quad \alpha=\int_0^\infty x\,dF(x) \quad \text{and} \quad \lambda=1/\alpha.$$

We suppose $\alpha<\infty$.

(ii) Units are served individually by a single server. Since the units of a batch arrive simultaneously, we shall suppose that they are ordered for purposes of service. Batches are served in order of arrival. Denote by χ_n the service time of the n^{th} unit to be served. We suppose that $\{\chi_n\}$ $(n=1,2,\ldots)$ is a sequence of identically distributed independent positive random variables, independent also of the sequence $\{\tau_n\}$, and that their common distribution function, $H(x)$, is the exponential distribution:

$$H(x)=\mathbf{P}[\chi_n\leqq x]=1-e^{-\mu x} \qquad (x\geqq 0).$$

Put $\beta=\int_0^\infty x\,dH(x)$. Then $\mu=1/\beta$. Define $\varrho=\lambda/\mu$.

In the terminology of [3], the system we consider has the 1-input (arrivals) untriggered with input quantity constantly r and a general distribution for the 1-input time. The 0-input (departures) is triggered with input quantity constantly unity and an exponential distribution for the 0-input time. The system has infinite capacity. On account of the characteristic property of the exponential distribution we have the alternative of supposing that the 0-input is untriggered also but with controlled input quantity: the input being virtual whenever the system contains no 1's. In other words, service begins from time to time whether or not there are any units in the system, and if at

the end of a service time a unit is present, it departs, otherwise nothing happens, and a fresh service begins.

Such batch-size queueing processes do not appear to have been treated explicitly in the literature, although they have obvious applications. They are, however, implicit in the work of ERLANG (see [1]) and WISHART [9]. These authors suppose that a service time devoted to one unit is composed of r consecutive phases. If, instead, we think of the unit as composed of r sub-units corresponding to the phases of service, we have the idea of batch arrivals. Justification for the explicit consideration of batch arrivals systems resides in the fact that the results one can obtain are elegant, and a natural generalization of the case of unit arrivals, as treated for example in [5] and [7]. This paper covers much the same ground as [9], but the analysis is different and the results obtained here are in fact new. (Cf. my remarks in the Discussion of [8].)

Denote by $\xi(t)$ the number of units in the system, including the one being served, at the instant t and put $\xi_n = \xi(\tau_n - 0)$ $(n = 1, 2, \ldots)$. The main result of this paper is the determination of the limiting distribution,

$$p_j = \lim_{n\to\infty} \mathbf{P}[\xi_n = j].$$

I am indebted to Dr. L. TAKÁCS for suggesting a substantial improvement in my original method of proof. The distribution $\{p_j\}$ exists and is independent of the initial state of the system if and only if $r\varrho < 1$. The proof of this statement follows the same lines as that for the case $r = 1$, as given in [2].

The limiting distribution of the waiting time for an arbitrary unit will also be derived.

2. Let $\{\nu_n\}$ $(n = 1, 2, \ldots)$ be a sequence of identically distributed independent random variables with distribution

$$k_j = \mathbf{P}[\nu_n = j] \qquad (j = 0, 1, 2, \ldots)$$

where

$$k_j = \int_0^\infty e^{-\mu x} \frac{(\mu x)^j}{j!} \, dF(x).$$

Then ν_n is thought of as the number of real or virtual departures during the n^{th} inter-arrival time.

Put $K(z) = \sum_{j=0}^\infty k_j z^j$. We note that $K(z) = \varphi\{\mu(1-z)\}$. We assume $r\varrho < 1$. Then it follows from Rouché's theorem that the equation

(1) $$K(z) = z^r$$

has exactly r roots (distinct or coincident) inside the circle $|z|=1$. For $K'(1)=-\mu\varphi'(0)=1/\varrho>r$, and so for some small positive δ, $K(1-\delta)<(1-\delta)^r$. Therefore, on the circle $|z|=1-\delta$, $|K(z)|\leq\Sigma k_j|z|^j<(1-\delta)^r=|z|^r$. Denote the roots by $z=\gamma_i$ $(j=1,2,\ldots,r)$. Clearly, one and only one of these roots is real and positive.

Define $P(z)=\sum_{j=0}^{\infty}p_j z^j$.

THEOREM 1.

$$P(z)=\prod_{j=1}^{r}\frac{1-\gamma_j}{1-\gamma_j z}, \tag{2}$$

the result being true whether or not some of the roots, γ_j, are coincident.

PROOF. By virtue of the characteristic property of the exponential distribution referred to in Section 1,

$$\xi_{n+1}=\max[\xi_n+r-\nu_n,0].$$

Letting $n\to\infty$ and taking generating functions, we have, for $|z|=1$,

$$P(z)=P(z)z^rK(z^{-1})+\sum_{j=0}^{\infty}c_j(1-z^{-j}),$$

where $\{c_j\}$ $(j=0,1,2,\ldots)$ is a sequence of real non-negative constants for which $\sum_{j=0}^{\infty}c_j=p_0$. Therefore

$$P(z)=\frac{\Sigma c_j(1-z^{-j})}{1-z^rK(z^{-1})}.$$

We now have to determine the constants c_j, and for this purpose we consider the zeros of the denominator. Clearly, there are exactly r zeros outside the unit circle, and these are $1/\gamma_j$ $(j=1,2,\ldots,r)$. Now consider the function

$$A(z)=P(z)\prod_{j=1}^{r}(1-\gamma_j z).$$

Since $P(z)$ is the generating function of a probability distribution, $A(z)$ must be regular for $|z|\leq 1$. Now let $A(z)$ be defined for $|z|>1$ by

$$A(z)=\frac{\prod_{j=1}^{r}(1-\gamma_j z)\sum_{j=0}^{\infty}c_j(1-z^{-j})}{1-z^rK(z^{-1})}.$$

Since in this expression all the zeros of the denominator outside the circle $|z|=1$ are also zeros of the numerator, it follows that $A(z)$ is regular for $|z|>1$. Therefore, by analytic continuity, $A(z)$ is defined and regular for

all z. Since, moreover, $A(z)=o(|z|)$ as $|z|\to\infty$, it follows that $A(z)=A$, a constant independent of z. Therefore

$$P(z)=\frac{A}{\prod_{j=1}^{r}(1-\gamma_j z)},$$

and from $P(1)=1$ we obtain finally (2).

We note that the distribution $\{p_j\}$ is thus formally the convolution of r geometric "distributions", if we allow complex probabilities. We have here a natural generalization of the known result for $r=1$ (see [5]).

EXAMPLE 1. *Inter-arrival times exponentially distributed.*

Suppose

$$F(x)=1-e^{-\lambda x} \qquad (x\geqq 0).$$

Then $\varphi(s)=\lambda/(\lambda+s)$. Therefore $K(z)=\varphi\{\mu(1-z)\}=\lambda/\{\lambda+\mu(1-z)\}=$ $=\varrho/(\varrho+1-z)$. Therefore the r roots γ_j are the roots of

$$\frac{\varrho}{\varrho+1-z}-z^r,$$

inside the circle $|z|=1$. This equation can be written,

$$(1-z)z^r+\varrho z^r=\varrho, \tag{3}$$

which, being a polynomial equation of degree $r+1$, has exactly $r+1$ roots, and one of them is seen to be $z=1$.

Now consider the zeros of the expression

$$(1-z)\prod_{j=1}^{r}(1-\gamma_j z).$$

They are

$$1,\gamma_1^{-1},\gamma_2^{-1},\ldots,\gamma_r^{-1}.$$

But these are the reciprocals of the roots of equation (3). Therefore they are the roots of the equation

$$(1-z^{-1})z^{-r}+\varrho z^{-r}=\varrho,$$

i.e.

$$1-z\{1+\varrho(1-z^r)\}=0. \tag{4}$$

Thus we have the identity

$$(1-z)\prod_{j=1}^{r}(1-\gamma_j z)\equiv 1-z\{1+\varrho(1-z^r)\}.$$

It follows that, in this case, we have

$$P(z) = \frac{(1-z)\prod_{j=1}^{r}(1-\gamma_j)}{1-z\{1+\varrho(1-z^r)\}}.$$

From $P(1)=1$ we obtain $\prod_{j=1}^{r}(1-\gamma_j)=1-r\varrho$, so that finally

$$P(z) = \frac{(1-r\varrho)(1-z)}{1-z\{1+\varrho(1-z^r)\}}. \tag{5}$$

Thus in the special case of exponential inter-arrival times, the generating function $P(z)$ can be expressed in a form not explicitly involving the roots γ_j.

EXAMPLE 2. *Inter-arrival times having an Erlang distribution.*

Suppose

$$F(x) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda k x)^j}{j!} e^{-\lambda k x}.$$

Then $\varphi(s) = \{\lambda k/(\lambda k+s)\}^k$, and so $K(z) = [\varrho/\{\varrho+(1-z)k^{-1}\}]^k$. The roots γ_j are thus the roots of

$$\varrho^k = z^r\{\varrho+(1-z)k^{-1}\}^k, \tag{6}$$

inside the circle $|z|=1$. This equation, being a polynomial equation of degree $r+k$, has $r+k$ roots. We know that $r+1$ of them are

$$1, \gamma_1, \gamma_2, \ldots, \gamma_r.$$

Let the other $k-1$ roots, which will be outside the circle $|z|=1$, be

$$\alpha_1, \alpha_2, \ldots, \alpha_{k-1}.$$

Now the equation whose roots are the reciprocals of the roots of (6) is

$$\varrho^k = z^{-r}\{\varrho+(1-z^{-1})k^{-1}\}^k,$$

i. e.

$$(k\varrho)^k z^{r+k} - \{(1+k\varrho)z-1\}^k = 0. \tag{7}$$

The left-hand side of this equation may therefore be identified with

$$(-1)^{k+1}(1-z)\prod_{j=1}^{r}(1-\gamma_j z)\prod_{j=1}^{k-1}(1-\alpha_j z).$$

Therefore we have, in this case,

$$P(z) = \frac{(-1)^{k+1}(1-z)\Pi(1-\alpha_j z)\Pi(1-\gamma_j)}{(k\varrho)^k z^{r+k} - \{(1+k\varrho)z-1\}^k}.$$

From $P(1)=1$ we obtain

$$\prod_{j=1}^{r}(1-\gamma_j)=\frac{(-1)^{k+1}(k\varrho)^{k-1}k(1-r\varrho)}{\prod_{j=1}^{k-1}(1-\alpha_j)},$$

so that finally

$$P(z)=\frac{(k\varrho)^{k-1}k(1-r\varrho)(1-z)}{(k\varrho)^k z^{r+k}-\{(1+k\varrho)z-1\}^k}\cdot\prod_{j=1}^{k-1}\frac{1-\alpha_j z}{1-\alpha_j}. \tag{8}$$

Thus in the case of Erlang inter-arrival times, the generating function $P(z)$ can be expressed in a form which involves explicitly only the roots α_j of (6), lying outside the unit circle, or equivalently the roots $1/\alpha_j$ of (7), lying inside the unit circle.

When k is odd, all the α_j are complex; when k is even, exactly one α_j is real and positive. This may be seen by a consideration of the pair of curves given by

$$y=(k\varrho)^k z^{r+k} \tag{9}$$

and

$$y=\{(1+k\varrho)z-1\}^k, \tag{10}$$

for real z. They intersect at $z=1$, and since $r\varrho<1$, the gradient at $z=1$ of (9) is less than that of (10). When k is even, the curves are seen to intersect at one other point lying between $z=0$ and $z=1$. But when k is odd, they do not intersect at any real value of z such that $|z|<1$.

For example, when $k=2$, there is only one α_i, say α, which must therefore be real and positive, and so we have, in this case,

$$P(z)=\frac{4\varrho(1-r\varrho)(1-z)}{4\varrho^2 z^{r+2}-\{(1+2\varrho)z-1\}^2}\,\frac{1-\alpha z}{1-\alpha}, \tag{11}$$

where α^{-1} is the unique real zero, lying within the interval $(0,1)$, of the denominator.

3. The waiting time distribution of an arbitrary arriving unit. We consider first the waiting time of the first unit in an arbitrary arriving batch. The waiting time is defined as the time which elapses between the instant at which the unit arrives and the instant at which its service begins. Denote by $\eta(t)$ the waiting time which the unit would have if it arrived at time t, and define $\eta_n=\eta(\tau_n-0)$. Thus η_n is the waiting time of the first unit in the n^{th} arriving batch. We consider the limiting distribution,

$$W(x)=\lim_{n\to\infty}\mathbf{P}\,[\eta_n\leq x].$$

Put

$$\Omega(s) = \int_0^\infty e^{-sx}\, d\, W(x).$$

THEOREM 2.

12) $$\Omega(s) = \prod_{j=1}^{r} \frac{1-\gamma_j}{1-\dfrac{\gamma_j \mu}{\mu+s}}.$$

PROOF. Each unit has a service time whose distribution is exponential with Laplace transform $\mu/(\mu+s)$. By the characteristic property of the exponential distribution, we can suppose that the service time of the unit at the head of the queue re-commences at the instant of the arrival of a batch. If an arriving batch finds j units in the system, the waiting time of the first unit in the batch will have Laplace transform $\{\mu/(\mu+s)\}^j$. The asymptotic probability of j units in the system is p_j. Therefore

$$\Omega(s) = \Sigma p_j \left(\frac{\mu}{\mu+s}\right)^j = P\left(\frac{\mu}{\mu+s}\right),$$

which gives the required result.

We note that, if we allow complex probabilities, the waiting time distribution is formally the convolution of r exponential "distributions", with concentrations $1-\gamma_j$ ($j=1, 2, \ldots, r$) at the origin.

COROLLARY. The waiting time distribution of a random unit in a batch has Laplace transform

(13) $$\prod_{j=1}^{r} \frac{1-\gamma_j}{1-\dfrac{\gamma_j \mu}{\mu+s}} \cdot \frac{1}{r} \sum_{j=0}^{r-1} \left(\frac{\mu}{\mu+s}\right)^j.$$

4. Relationship with the unit arrivals system $G/E_r/1$. Let us now consider that the batches retain their identity in the queue: a batch is being served until its last member departs. Then if ξ is the number of units facing an arriving batch, the number of batches facing an arriving batch will be ζ where

$$\zeta = \begin{cases} 0 & \text{if} \quad \xi = 0, \\ \left[\dfrac{\xi-1}{r}\right] + 1 & \text{if} \quad \xi \neq 0, \end{cases}$$

where $[x]$ denotes the greatest integer not greater than x. We may now interpret the random variable ζ as the number of units facing an arbitrary arriving unit in the unit arrivals system, $G/E_r/1$, which has mean inter-arrival time

$1/\lambda$, and an Erlang service time distribution with a mean of r/μ. The traffic intensity is thus $r\varrho$.

We consider the distribution of ζ. Define

$$q_j = \mathbf{P}[\zeta = j],$$

and put $Q(z) = \sum_{j=0}^{\infty} q_j z^j$. Now define

$$P_j = \sum_{i=0}^{j} p_i \quad \text{and} \quad Q_j = \sum_{i=0}^{j} q_i .$$

Then

$$\sum_{j=0}^{\infty} P_j z^j = \frac{P(z)}{1-z} \quad \text{and} \quad \sum_{j=0}^{\infty} Q_j z^j = \frac{Q(z)}{1-z} .$$

We have

$$Q_0 = P_0, \quad Q_1 = P_r, \quad Q_2 = P_{2r},$$

and generally,

$$Q_j = P_{jr} .$$

Therefore

$$\frac{Q(z)}{1-z} = \sum_{j=0}^{\infty} P_{jr} z^j .$$

But

$$P_j = \frac{1}{2\pi i} \int_C \frac{P(v)}{1-v} \frac{dv}{v^{j+1}} ,$$

where C is a contour around the origin excluding the poles of $P(z)\,(1-z)$. Therefore

$$\frac{Q(z)}{1-z} = \sum_{j=0}^{\infty} \frac{z^j}{2\pi i} \int_C \frac{P(v)}{1-v} \frac{dv}{v^{j+1}} ,$$

so that

(14) $$Q(z) = \frac{1-z}{2\pi i} \int_C \frac{P(v)}{v(1-v)} \frac{dv}{(1-v^{-r} z)} .$$

The poles of the integrand within C are at

$$v = \omega^j z^{1/r} \qquad (j = 1, 2, \ldots, r)$$

where ω^j is an r^{th} root of unity. The residue at $v = \omega^j z^{1/r}$ is

$$\frac{1}{r} \frac{P(\omega^j z^{1/r})}{1-\omega^j z^{1/r}} .$$

Therefore, summing the residues, we obtain

$$Q(z) = \frac{1-z}{r} \sum_{j=1}^{r} \frac{P(\omega^j z^{1/r})}{1-\omega^j z^{1/r}} \quad . \tag{15}$$

We may note that the waiting time distribution for an arbitrary arriving unit in the unit arrivals system $G/E_r/1$ is identical with that for an arbitrary arriving batch in the batch arrivals system, and so is given by formula (12) above. Thus we have a solution in an interesting form to the Wiener—Hopf integral equation studied by LINDLEY [6].

EXAMPLE 3. *Inter-arrival times exponentially distributed.*

The generating function $P(z)$ for the batch arrivals system with exponential inter-arrival times is given by (5). Therefore if we consider the unit arrivals system $M/E_r/1$ with mean inter-arrival time $1/\lambda$ and mean service time r/μ, the generating function $Q(z)$ for the queue-size facing an arbitrary arriving unit is given by

$$\frac{Q(z)}{1-z} = \frac{1-r\varrho}{r} \sum_{j=1}^{r} \frac{1}{1-\omega^j z^{1/r}\{1+\varrho(1-z)\}} .$$

But the $\omega^j z^{1/r}\{1+\varrho(1-z)\}$ $(j=1,2,\ldots,r)$ are the r^{th} roots of $z\{1+\varrho(1-z)\}$. Therefore

$$Q(z) = \frac{(1-r\varrho)(1-z)}{1-z\{1+\varrho(1-z)\}^r} , \tag{16}$$

which is the known result for the system $M/E_r/1$ when the traffic intensity is $r\varrho$.

The waiting time distribution of an arbitrary arriving unit has in this case Laplace transform

$$\Omega(s) = P\left(\frac{\mu}{\mu+s}\right) = \frac{(1-r\varrho)\dfrac{s}{\mu+s}}{1-\dfrac{\mu}{\mu+s}\left\{1+\varrho-\varrho\left(\dfrac{\mu}{\mu+s}\right)^r\right\}} = \frac{1-r\varrho}{1-\dfrac{\lambda}{s}\left\{1-\left(\dfrac{\mu}{\mu+s}\right)^r\right\}},$$

which is the known Pollaczek formula for this case.

5. Further work. Let $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ denote the sequence of instants at which units depart from the batch arrivals system. In a sequel we shall consider the existence of the limiting distributions $\{p_j^*\}$ and $\{p_j^+\}$, defined, respectively, by

$$p_j^* = \lim_{t\to\infty} \mathbf{P}[\zeta(t)=j]$$

and

$$p_j^+ = \lim_{n\to\infty} \mathbf{P}[\xi(\sigma_n+)=j].$$

We shall also examine the relationships existing between the three distributions $\{p_j\}$, $\{p_j^*\}$ and $\{p_j^+\}$.

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ON PERIODIC SOLUTIONS OF CERTAIN SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

By

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(*Presented by* P. TURÁN)

As known, the investigation of the existence of periodic solutions of linear equations with periodic coefficients (e. g. the Hill and Mathieu equations) involves considerable amount of difficulties. The Floquet theory discussing this questions cannot be considered elementary at all. As to non-linear equations (e. g. the Liénard equation) the problem is still much more involved. N. LEVINSON, M. L. CARTWRIGHT, T. WAZEWSKI and other authors elaborated intricate analytical and analytic-topological methods to this end. On the other hand, the proof of the existence of such solutions is often very desirable for the practice too, for a solution like this is connected with some kind of stability (e. g. it forms a limit cycle in the sense of the Poincaré—Bendixson theory).

Even therefore it is surprising how simple tools lead to results concerning the equation

$$y''+\varphi(x)f(y)h(y')=0, \tag{1}$$

provided that $\varphi(x)$ is a *positive periodic* function, $f(y)$ and $h(u)$ are like those in [1] (p. 98, Theorems 5 and 6) where — as throughout [1] — only *positive monotone* $\varphi(x)$ was taken into account. The purpose of the present paper is to state results of this character, of course, also for linear equations.

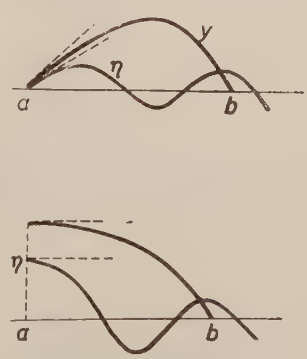

Fig. 1

On p. 102 of [1] a remark says that there is a solution $\eta(x)$ of (1) to any other one $y(x)$ of (1) with an arbitrary number of zeros between two adjacent zeros of $y(x)$ and $\eta(x)$ can be obtained by a convenient (sufficiently small) initial slope or (extreme) value η of $\eta(x)$ at $x=a$ (see Fig. 1). — This assertion must be corrected, for Corollary 8 on p. 87 in [1] does not imply really, as asserted, that the distance of two consecutive zeros tends to zero with the above initial values (see a counterexample below). In case of $\varphi(x)=k=\text{const}>0$ there is given ([1], p. 88) a formula for this distance (i. e. for the period p, viz.

the corresponding solution is periodic) and it is doubtless that p is decreasing with $|\eta|$ (at linear equations unchanged) (see [1], Theorem 5), but, in general, does not tend to zero with $|\eta|$ or by the initial slope.

To throw light on the things take first the following example:

$$y'' + \varphi(x)y\frac{1}{1+\varepsilon^2 y'^2} = 0. \tag{2}$$

Here $f(y) = y$, $h(u) = \frac{1}{1+\varepsilon^2 u^2}$ satisfy the mentioned conditions and equation (2) turns to the linear equation $y'' + \varphi(x)y = 0$ as $\varepsilon \to 0$. If $\varphi(x) = k = \text{const}$, (2) may be solved by quadratures. According to $y(0) = \eta > 0$, $y'(0) = 0$, the solution is as follows (see [1], p. 89):

$$\varepsilon \int_y^{\eta} \frac{du}{\sqrt{-1+\sqrt{1+2\varepsilon^2 k(\eta^2-u^2)}}} = x \qquad (\varepsilon > 0) \tag{3}$$

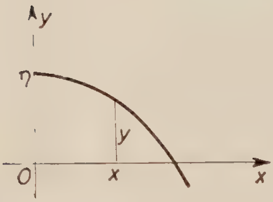

Fig. 2

and the period of $y(x)$ (see [1], p. 87) amounts to

$$p(\varepsilon, \eta, k) = 4\varepsilon \int_0^{\eta} \frac{du}{\sqrt{-1+\sqrt{1+2\varepsilon^2 k(\eta^2-u^2)}}} =$$

$$= \frac{2}{\sqrt{k}} \int_0^{\eta} \frac{\sqrt{1+\sqrt{1+2\varepsilon^2 k(\eta^2-u^2)}}}{\sqrt{\eta^2-u^2}}\, du.$$

Replacing u by $\eta \sin z$ we get

$$p(\varepsilon, \eta, k) = \frac{4}{\sqrt{2k}} \int_0^{\frac{\pi}{2}} \sqrt{1+\sqrt{1+2\varepsilon^2 k \eta^2 \cos^2 z}}\, dz. \tag{4}$$

This value may be made as large as wanted by increasing of $|\eta|$, provided $\varepsilon \neq 0$. On the other hand, p is decreasing with $|\eta|$ and we have

$$p(\varepsilon, 0, k) = \frac{2\pi}{\sqrt{k}} = p(0, \eta, k) = p(0, 0, k),$$

i. e. p tends with $|\eta| \to 0$ to the period of the linear equation $y'' + ky = 0$.

We shall prove the following

THEOREM 1. *Let $\varphi(x)$ be a positive continuous periodic function having period T and monotone symmetrical half-periods, further let* $\min \varphi(x)$ *and*

$\max \varphi(x)$ *be denoted by* k *and* K, *respectively. If* $kT^2 \geqq n^2\pi^2$,[1] *then for* $n=1$ (2) *has a periodic solution of period* $2T$ *consisting of two half-waves, while for* $n = n_0 > 1$ (2) *has all the solutions as for* $n < n_0$ *and also a new one of period* $2T$ *consisting of* $2n_0$ *half-waves, provided that* $n_0 = 2l+1$ ($l = 1, 2, \ldots$), *or a solution of period* T *consisting of* n_0 *half-waves, provided that* $n_0 = 2l$ ($l = 1, 2, \ldots$).

PROOF. Let $\varphi(x)$ be, say, increasing for $0 \leqq x \leqq \frac{T}{2}$, then the solutions y_1, y_2 of the equations

$$y'' + ky\frac{1}{1+\varepsilon^2 y'^2} = 0, \quad y'' + Ky\frac{1}{1+\varepsilon^2 y'^2} = 0$$

and the solution y of (2), all corresponding to the initial conditions $y(0) = \eta$, $y'(0) = 0$, satisfy the inequalities (see [1], p. 102)

$$y_2 < y < y_1 \qquad (0 < x \leqq a),$$
$$y < y_1 \qquad (0 < x \leqq b).$$

Here $a < b < c$ denote the first positive zeros of y_2, y, y_1, respectively (see Fig. 3) and $b \leqq \frac{T}{2}$ is assumed too. For the value of c we have

$$c(\varepsilon, \eta, k) = \frac{p(\varepsilon, \eta, k)}{4}$$

(see (4)). Obviously

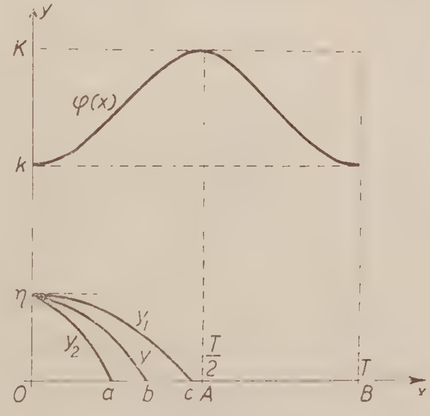

Fig. 3

$$c(\varepsilon, 0, k) = \frac{\pi}{2\sqrt{k}} \qquad \text{and} \qquad c(\varepsilon, \infty, k) = +\infty$$

and c is a monotone increasing function of $|\eta|$. Therefore assuming

(5) $$c(\varepsilon, 0, k) = \frac{\pi}{2\sqrt{k}} \leqq \frac{T}{2} \quad \text{or} \quad kT^2 \geqq \pi^2,$$

there is a value $\eta_0 > 0$ of η where $b(\varepsilon, \eta_0) = \frac{T}{2}$, because the interval $[a, c]$, with the point $x = b$ in its interior, passes wholly through the point $A\left(\frac{T}{2}, 0\right)$ as η covers $(0, +\infty)$. At the same time, exactly one quarter-wave of $y(x)$

[1] n denotes an integer.

gets into the interval $0 \leqq x \leqq \frac{T}{2}$. We state that $y(x)$ is periodic with period $2T$. Viz. the symmetrical $\eta(x)$ of $y(x)$ $\left(0 \leqq x \leqq \frac{T}{2}\right)$ with respect to the point A satisfies (2) and thus it is the continuation of $y(x)$ for $\frac{T}{2} \leqq x \leqq T$. Really, putting $T-x$ in (2) for x and regarding the relations

$$\eta(x) = -y(T-x), \quad \eta'(x) = y'(T-x),$$
$$\eta''(x) = -y''(T-x),$$
$$\varphi(T-x) = \varphi(-x) = \varphi(x),$$

we obtain

$$\eta''(x) + \varphi(x)\eta(x)\frac{1}{1+\varepsilon^2\eta'^2(x)} = 0.$$

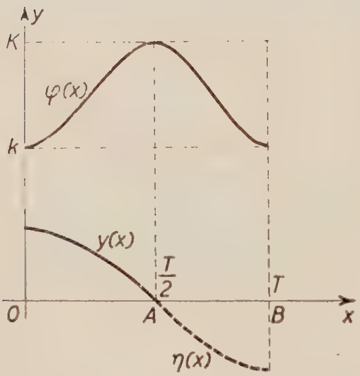

Fig. 4

It can be proved by the same argument that the continuation of $y(x)$ for $T \leqq x \leqq 2T$ is symmetrical with the branch $0 \leqq x \leqq T$ of $y(x)$ to the ordinate $x = T$, etc., thus the periodicity of $y(x)$ by the period $2T$ is clear.

If we assume instead of (5) the condition

$$c(\varepsilon, 0, k) = \frac{\pi}{2\sqrt{k}} \leqq \frac{T}{2n} \tag{6}$$
$$\text{or} \quad kT^2 \geqq n^2\pi^2 \qquad (n \geqq 2),$$

then by a convenient choice of η we obtain also a new solution $y(x)$ of period T or $2T$, according as $n = 2l$ or $n = 2l+1$, respectively.[2] Viz. for a suitable η exactly n quarter-waves of $y(x)$ will lie in the interval $0 \leqq x \leqq \frac{T}{2}$ with $y'\left(\frac{T}{2}\right) = 0$ or $y\left(\frac{T}{2}\right) = 0$, respectively. This is really ensured by the condition (6), because the monotone increasing of $\varphi(x)$ involves the decreasing of the lengths, amplitudes, areas of the successive quarter-waves (see [1], Theorem 6). The branch $\frac{T}{2} \leqq x \leqq T$ of

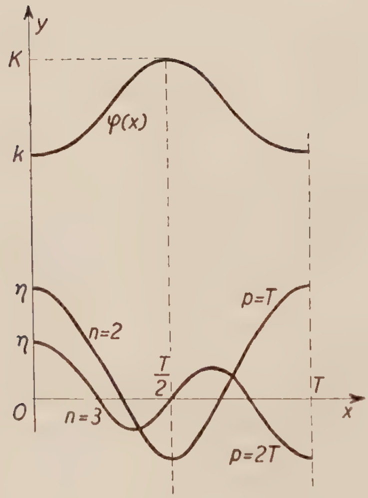

Fig. 5

[2] If (6) is satisfied for $n = n_0$, then it is satisfied for $n < n_0$, too, and the corresponding solutions exist also for $n = n_0$.

$y(x)$ is symmetrical with its own $0 \leqq x \leqq \frac{T}{2}$ branch to the *ordinate* $x = \frac{T}{2}$ or to the *point* $x = \frac{T}{2}$, respectively, etc.

The above statements are concerned with the non-linear case $\varepsilon \neq 0$. For every $\varepsilon \neq 0$ there are periodic solutions, provided that (6) is fulfilled. Let us denote the above periodic solution of period $2T$ for $n = 1$ by $y(x, \varepsilon, \eta)$. If $\varepsilon \to 0$ and (5) holds, then $\eta_0 \to +\infty$, i. e. there is no periodic solution of period $2T$ of the linear equation $(\varepsilon = 0)$

$$y'' + \varphi(x) y = 0, \tag{7}$$

unless $b = \frac{T}{2}$ holds in advance. This is obvious by the fact, too, that the first positive zero of $y(x, 0, \eta)$ (viz. $x = b$) is independent of η.

At all events the condition

$$\frac{\pi}{\sqrt{K}} < T < \frac{\pi}{\sqrt{k}}$$

forms a necessary one for the existence of the above solution.

Although the limit passage $\varepsilon \to 0$ does not lead to periodic solutions, however, we obtain the following — only in part known — result concerning the linear equation

$$y'' + (\alpha + \beta\varphi(x)) y = 0. \tag{8}$$

THEOREM 2. *Let $\varphi(x)$ be a continuous periodic function of period T with monotone symmetrical half-periods. Then α and β may be chosen so that* (8) *should have periodic solutions like those described in Theorem 1.*

PROOF. Let $\varphi(x)$ be e. g. even increasing for $0 \leqq x \leqq \frac{T}{2}$ and letting $k = \alpha + \beta\varphi(0)$, $K = \alpha + \beta\varphi\left(\frac{T}{2}\right)$ we have

$$k = \min(\alpha + \beta\varphi(x)), \quad K = \max(\alpha + \beta\varphi(x)) \qquad (\beta > 0).$$

Given $\eta \neq 0$ let y_2, y_1, y denote the solutions of the equations

$$y'' + Ky = 0, \quad y'' + ky = 0$$

and of (8) — all corresponding to the initial conditions $y(0) = \eta$, $y'(0) = 0$ — and let $a < b < c$ be the first positive zeros of y_2, y, y_1, respectively.

If $K = g(k) > k > 0$ where $g(k)$ is an arbitrary continuous monotone function with $g(k) \to 0$ as $k \to 0$,[3] then the interval $[a, c]$ (Fig. 3), with the

[3] Or more generally $g(0) \leqq \frac{\pi^2}{T^2}$.

point $x=b$ in its interior, passes through the place $x=\frac{T}{2}$ as $k\to 0$. Therefore there exists a value $k=k_0>0$ of k such that $b=\frac{T}{2}$ is satisfied. Then

$$\alpha=\frac{k_0\varphi\left(\frac{T}{2}\right)-K_0\varphi(0)}{\varphi\left(\frac{T}{2}\right)-\varphi(0)},\qquad \beta=\frac{K_0-k_0}{\varphi\left(\frac{T}{2}\right)-\varphi(0)}\qquad (K_0=g(k_0))$$

and $y(x)$ is periodic with period $2T$, etc.

If in the general equation (1) $\varphi(x)=k=\text{const}>0$, the solution is periodic with the period

$$p=4\int_0^{\eta}\frac{du}{H^{-1}(k[F(\eta)-F(u)])}\qquad \left(F(y)=\int_0^{y}f(u)\,du,\quad H(u)=\int_0^{u}\frac{z\,dz}{h(z)}\right).$$

This is increasing with $|\eta|$ (see [1], Theorem 5) and has a zero or positive limit as $\eta\to 0$ and a finite or infinite limit as $\eta\to\infty$. Denoting these limits by $p(0,k)$ and $p(\infty,k)$ we can state

THEOREM 3. *Assuming*

$$\frac{p(0,k)}{2}\leqq\frac{T}{n}\leqq\frac{p(\infty,K)}{2},$$

equation (1) *has for convenient* η *solutions of period* $2T$ *and* T *like above in Theorem 1. — The conditions imposed on* $\varphi(x), f(y), h(u)$ *are the same as in Theorem 5 in* [1].

The proof follows previous lines and may be omitted.

Equation (1) is that particular case of the general equation $y''=f(x,y,y')$ where $f(x,y,y')$ is factorized. However, the analogue of Theorem 2 may be extended to the solutions of the equation

(9) $$y''+(\alpha+\beta\varphi(x))f(y,y')=0$$

where $f(\lambda u,\lambda v)=\lambda f(u,v)$ and $\operatorname{sg} f(u,v)=\operatorname{sg} u$. This equation — discussed in a paper [2] of the author — shows common features with the linear equations.

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STANDARD IDEALS IN LATTICES

By

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(*Presented by* A. Rényi)

To Professor Ladislaus Fuchs *without whose constant help and encouragement we could never have achieved our humble results*

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Introduction

The subject of this paper is to define a special class of lattice ideals, the class of standard ideals, and to examine its properties in detail. Before giving the definition of standard ideal we want the reader make acquainted with three tendencies of modern lattice theory which lead naturally to this notion.

The distributive lattices play a central role in lattice theory. This may be explained, on one hand, by the fact that lattices were abstracted from Boolean algebras through the distributive lattices. On the other hand, the distributive lattices have a lot of important properties that lattices in general do not have and, consequently, many of the researches were restricted to distributive lattices.

This fact gives the reason why some mathematicians have tried to define types of elements resp. ideals of lattices which preserve some properties of distributive lattices. It was of importance when in his paper [23] O. ORE has defined the notion of neutral element and ideal in modular lattices, and it was also of significance that in [4] G. BIRKHOFF succeeded in defining these notions in arbitrary lattices. The neutral elements play a central role, for instance, in the theory of direct factorizations of lattices (see [6]). Therefore the question how it is possible to generalize this notion to a wider class of lattice elements and ideals seems to be of interest.

Another trend of researches wants to elaborate the theory of lattice ideals similarly to the theory of ideals in rings or invariant subgroups in groups. Chiefly we are thinking of the fact that any ideal of a ring is the kernel of one and only one homomorphism, furthermore the ideals satisfy the well-known isomorphism theorems, the lemma of Zassenhaus and the Jordan—Hölder—Schreier refinement theorem. Such efforts have to overbridge many difficulties. Naturally, within the Boolean algebras — since the Boolean algebras are rings as well — the researcher does not meet any difficulty. It is also easy to settle this question in distributive lattices, only a good definition of the factor lattice is needed. (The simplest possible method is the following: we embed the distributive lattice in a Boolean algebra — for instance by the method of [13] — and so we get from the well-known notions and theorems of Boolean algebras the same in distributive lattices.)

The case of general lattices is not so simple. In general, the above mentioned theorems are not true. In his paper [31] K. SHODA avoided these difficulties by a suitable definition of the factor algebra; this definition of factor algebra, however, in case of lattices does not seem to be applicable. This was pointed out in [14] by J. HASHIMOTO, remarking that this definition of factor algebra in chains gives only the factor chain of two elements.

In [14], using an other definition of factor lattices, J. Hashimoto has proved interesting isomorphism theorems. Hashimoto made the very strong restriction: all the ideals occurring in the isomorphism theorems, are neutral. The question arises: is it possible to enlarge the class of neutral ideals, preserving the validity of the isomorphism theorems?

The third tendency of researches that we are going to sketch has started from the Birkhoff—Menger structure theorem of complemented modular lattices of finite length (see G. Birkhoff [2], [3] and K. Menger [22]). This structure theorem asserts that the lattices of the above type coincide with the direct products of simple lattices. A theorem of R. P. Dilworth [8] states that this structure theorem remains true without any alteration if we omit the supposition of modularity (of course, we must change the word "complemented" to "relatively complemented"). In fact, with this theorem began the investigation of the structure of relatively complemented lattices. The aim of these researches is to prove the results of the theory of relatively complemented modular lattices for relatively complemented lattices as well (some example from among these kinds of papers: J. E. McLaughlin [20], [21] and G. Szász [32]).

The following theorem of G. Birkhoff [6] is well known: in a complemented modular lattice if we let a congruence relation Θ correspond to the ideal of all x with $x=0$ (Θ), then we get a natural one-to-one correspondence between neutral ideals and congruence relations. A theorem of Shih-Chiang Wang [34], connected with this theorem of G. Birkhoff, asserts that the lattice of all congruence relations of a complemented modular lattice is a Boolean algebra if and only if all neutral ideals are principal. However, if we want to formulate these theorems for relatively complemented lattices or for section complemented lattices (i. e. in which the intervals $[0, a]$, as lattices, are complemented), then we do not get in general true assertions. So the question arises, how it is possible to get natural generalizations of these theorems for relatively complemented lattices, i. e. one may ask for the class of ideals, that plays, from the point of view of homomorphisms, a similar role in relatively complemented (section complemented) lattices, as the neutral ideal in complemented modular lattices.

We see that the developments of these there tendencies of lattice theory raise a common request, namely, that of finding appropriate generalizations of neutral ideals, of course, one generalization to each tendency! It was a great surprise to us, when it became clear that the *very same generalization* of neutral ideals answers all the questions raised above. This generalization is given by the notion of standard element and ideal.

An element s of the lattice L will be called *standard* if

$$x \cap (s \cup y) = (x \cap s) \cup (x \cap y)$$

for all pairs of elements x, y of L. A standard ideal of L is defined as a standard element of the lattice of all ideals of L.

The aim of the present paper is to study the most important properties of the standard elements and ideals and, as an application, to prove that the standard ideals make us possible to develop further the above listed three tendencies of lattice theory. We will prove that in some respects the notion of standard ideals is the best-possible one. Namely, the class of standard ideals is the widest one, satisfying the first isomorphism theorem, provided some natural conditions are assumed. Many other properties are also typical to the standard ideals, e. g. the existence of a "dictionary" — as given below. But, of course, if somebody will try to develop a theory of certain type of ideals, satisfying the requirements only of one of the above mentioned tendencies, then he will go further at the direction than we did.

It will appear from this paper that the notion of standard ideal corresponds to the notion of invariant subgroup of groups. Several theorems of group theory may be "translated" to lattice theory using the following "dictionary":[1]

subgroup → ideal

invariant subgroup → standard ideal

factor group → factor lattice[2]

group operation → join operation.[3]

We will use this "dictionary" for getting the appropriate forms of the isomorphism theorems, the Zassenhaus lemma, the solution of Schreier's extension problem and so on. We will see that the "dictionary" works well in all these cases. We get, of course, only the translations of the theorems but not those of the proofs!

The dictionary may be used also for translating negative assertions. An example: the invariant subgroup of an invariant subgroup is in general

[1] The "dictionary" may be used only in translating from group theory to lattice theory but not in the reversed direction! Therefore we used the sign → instead of equality.

[2] modulo a standard ideal!

[3] In the colloquium on Partially Ordered Sets (Oberwolfach, 26—30 October 1959) we have delivered a lecture in which a sketch of this theory was given. After the lecture Professor R. H. Bruck proposed an extension of the dictionary, that — after a short discussion — led to the correspondence

abelian group → distributive lattice.

Using this, one can define the solvability of a lattice, notions corresponding to the centralizator, and commutator subgroup and so on. It may be hoped that one can elaborate this part of the theory.

not invariant in the whole group and the same is true for standard ideals. (It is worth while mentioning that the neutral ideal of a neutral ideal is neutral in the whole lattice.)

Despite the fact that the notion of standard ideal is more general than that of neutral ideal, there appeared a lot of new properties of neutral ideals from the study of this generalization. Besides many not all too important properties, the best example is the result of Chapter VI (Theorem 23). This theorem characterizes neutral ideals in a special class of modular lattices. However, the proof shows clearly that the assertion is a typical one for standard ideals. Hence, we may say, that in this theorem we use the standard ideals as a method of proof.

The paper consists of six chapters.

The first chapter is of preliminary character. It contains notions which are not generally known, while for the fundamental notions of lattice theory and general algebra we refer to [6], [16] and [29]. The frequently used notions and theorems from the literature are enumerated.

In Chapter II, after the definition of standard element and ideal, we prove the two fundamental characterization theorems. In the remaining part we deduce some properties of the standard element and ideal which seems to be of importance.

In Chapter III we are interested in the connections between standard and neutral elements. In § 1 we verify the simplest connections, but already from these we deduce a new proof of a theorem concerning neutral ideals; a proof of this theorem within the theory of neutral ideals does not seem to be an easy task. In § 2 we prove the coincidence of standard and neutral elements in a rather wide class of lattices including modular as well as relatively complemented lattices. In § 3 we give a necessary and sufficient condition for a standard element to be neutral. In § 4 we deal with the lattice of all ideals of a weakly modular lattice. We prove that the lattice of all ideals is not necessarily weakly modular. In the remaining part of the section we discuss some properties of the ideal lattice.

In Chapter IV we prove that the class of standard ideals and that of the homomorphism kernels coincide in section complemented lattices. From this we infer the generalizations of the above mentioned theorems of G. Birkhoff and S. Wang. Then we prove the isomorphism theorems, the lemma of Zassenhaus and some of its consequences. In the last section we solve the lattice-theoretical equivalent of Schreier's extension problem.

In Chapter V we first prove that any distributive equality is capable of the characterization of the neutrality of an element of a modular lattice. Then in § 2 we prove that in modular lattices the uniquely relatively complemented

elements are just the neutral ones, and thus we get a generalization of a well-known theorem of VON NEUMANN.

In Chapter VI we deal with ideals satisfying the first isomorphism theorem. In § 1 for a special class of section complemented lattices, while in § 3 for modular lattices with zero and of locally finite length we prove that this class of ideals coincides with the class of neutral ideals. In § 4 we show that under some natural conditions the standard ideals form the widest class of ideals satisfying the first isomorphism theorem.

There are 20 unsolved problems given at the end of the corresponding sections. We hope some of the readers will find it interesting to deal with them.

CHAPTER I
PRELIMINARIES

§ 1. Some notions and notations

The partial ordering relation will be denoted by $\leq$, in case of set lattices (that is lattices the elements of which are certain subsets of a given set) by $\subset$. In lattices the meet and the join will be designated by $\cap$ and $\cup$, and the complete meet and complete join by $\bigwedge$ and $\bigvee$. The least and greatest element of a partially ordered set (or of a lattice) we denote by 0 and 1. If a covers b (i. e. $a > b$, but $a > x > b$ for no x), then we write $a \succ b$.

If $\alpha(x)$ is a property defined on the set H, then we define $\{x; \alpha(x)\}$ as the set of all $x \in H$ for which $\alpha(x)$ is true. Hence in partially ordered sets $(a] = \{x; x \leqq a\}$ is the principal ideal generated by a, while $\{x; a \leqq x \leqq b\}$ is the interval $[a, b]$, provided that $a \leq b$. If b covers a, then the interval $[a, b]$ is a prime interval. The dual principal ideal is denoted by $[a)$.

If any two elements a, b of L, satisfying $a < b$, may be connected by a finite maximal chain, then L is said to be semi-discrete. If the lengths of the maximal chains of the lattice L are finite and bounded, then L is called of finite length. If all intervals of the lattice L, as lattices, are of finite length, then L is of locally finite length. If L has a 0 and is of locally finite length, furthermore for all $a \in L$, in $[0, a]$ any two maximal chains are of the same length, then we say that in L the Jordan—Dedekind chain condition is satisfied. In this case the length of any maximal chain of the interval $[0, a]$ will be denoted by $d(a)$, and $d(x)$ is called the dimension function.

Let P and Q be partially ordered sets. The ordinal sum of P and Q is defined as the partially ordered set, which is the set union of P and Q,

and the partial ordering remains unaltered in P and Q, while $x<y$ holds for all $x \in P$ and $y \in Q$; this partially ordered set will be denoted by $P \oplus Q$.

The set of all ideals of a lattice L, partially ordered under set inclusion, form a lattice, which will be denoted by $I(L)$.

LEMMA 1. *$I(L)$ is a conditionally complete lattice. The meet of a set of ideals (if it exists) is the set-theoretical meet. The join of the ideals I_α ($\alpha \in A$) is the set of all x such that*

$$x \leqq i_{\alpha_1} \cup \cdots \cup i_{\alpha_n} \quad (i_{\alpha_j} \in I_{\alpha_j})$$

for some elements α_j of A.

If A is a general algebra and Θ is a congruence relation of A, then the congruence classes of A modulo Θ form a general algebra $A(\Theta)$. This is a homomorphic image of A.

We will use the two general isomorphism theorems (RÉDEI [29]):

THE FIRST GENERAL ISOMORPHISM THEOREM. *Let A be a general algebra and A' a subalgebra of A, further let Θ be an equivalence relation of A such that every equivalence class of A may be represented by an element of A'. Let Θ' denote the equivalence relation of A' induced by Θ. If Θ is a congruence relation, then so is Θ' and*

$$A(\Theta) \simeq A'(\Theta').$$

The natural isomorphism makes a congruence class of A correspond to the contained congruence class of A'.

THE SECOND GENERAL ISOMORPHISM THEOREM. *Let A' be a homomorphic image of the general algebra A, let Θ be an equivalence relation of A, and denote Θ' the equivalence relation of A' under which the equivalence classes are the homomorphic images of those of A modulo Θ, and suppose that no two different equivalence classes of A modulo Θ have the same homomorphic image. Then Θ is a congruence relation if and only if Θ' is one and in this case*

$$A(\Theta) \simeq A'(\Theta').$$

The natural isomorphism makes an equivalence class of A correspond to its homomorphic image.

§ 2. Congruence relations in lattices

Let Θ be a congruence relation of the lattice L, and denote by $L(\Theta)$ the homomorphic image of L induced by the congruence relation Θ, that is, the lattice of all congruence classes. If $L(\Theta)$ has a zero, then the complete

inverse image of the zero is an ideal of L, called the kernel of the homomorphism $L \to L(\Theta)$.

A simple criterion for a binary relation η to be a congruence relation is formulated in

LEMMA II. (GRÄTZER and SCHMIDT [12].) *Let η be a binary relation defined on the lattice L. η is a congruence relation if and only if the following conditions hold for all $x, y, z \in L$:*

(a) $x \equiv x\ (\eta)$;

(b) $x \cup y \equiv x \cap y\ (\eta)$ *if and only if* $x \equiv y\ (\eta)$;

(c) $x \geqq y \geqq z$, $x \equiv y\ (\eta)$, $y \equiv z\ (\eta)$ *imply* $x \equiv z\ (\eta)$;

(d) $x \geqq y$ *and* $x \equiv y\ (\eta)$, *then* $x \cup z \equiv y \cup z\ (\eta)$ *and* $x \cap z \equiv y \cap z\ (\eta)$.

The congruence relations of L will be denoted by $\Theta, \Phi, \ldots$. The set of all congruence relations of L, partially ordered by "$\Theta \leqq \Phi$ if and only if $x \equiv y\ (\Theta)$ implies $x \equiv y\ (\Phi)$", will be designated by $\Theta(L)$.

LEMMA III. (BIRKHOFF [4] and KRISHNAN [18].)[4] *$\Theta(L)$ is a complete lattice. $x \equiv y\ (\bigwedge_{\alpha \in A} \Theta_\alpha)$ if and only if $x \equiv y\ (\Theta_\alpha)$ for all $\alpha \in A$; $x \equiv y\ (\bigvee_{\alpha \in A} \Theta_\alpha)$ if and only if there exists in L a sequence of elements $x \cup y = z_0 \geqq z_1 \geqq \cdots \geqq z_n = x \cap y$ such that $z_i \equiv z_{i-1}\ (\Theta_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ for suitable $\alpha_1, \ldots, \alpha_n \in A$.*

The least and greatest elements of the lattice $\Theta(L)$ will be designated by ω and ι, respectively.

Let H be a subset of L, $\Theta[H]$ will denote the least congruence relation under which any pair of elements of H is congruent. This we call the congruence relation induced by H. If H has just two elements, $H = \{a, b\}$, then $\Theta[H]$ will be written as Θ_{ab}. The congruence relation Θ_{ab} is called minimal.

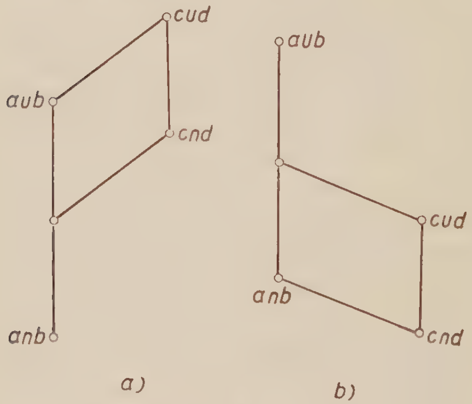

Fig. 1

First we describe — following R. P. DILWORTH — the minimal congruence relation Θ_{ab}. To this end we have to make some preparations.

Given two pairs of elements a, b and c, d of L, suppose that either

$$c \cap d \geqq a \cap b$$

and

$$(c \cap d) \cup (a \cup b) = c \cup d,$$

[4] See also GRÄTZER and SCHMIDT [12].

or

$$c \cup d \leqq a \cup b \quad \text{and} \quad (c \cup d) \cap (a \cap b) = c \cap d.$$

Then we say that a, b is weakly projective in one step to c, d, and write $\overline{a, b} \xrightarrow{1} \overline{c, d}$. The situation is given in Fig. 1. In other words, $\overline{a, b} \xrightarrow{1} c, d$ if and only if the intervals $[(a \cup b) \cap c \cap d, a \cup b]$, $[c \cap d, c \cup d]$ or $[a \cap b, (a \cap b) \cup c \cup d]$, $[c \cap d, c \cup d]$ are transposes in the sense of [6].

If there exist two finite sequences of elements $a = x_0, x_1, \ldots, x_n = c$ and $b = y_0, \ldots, y_n = d$ in L such that

$$(1) \qquad \overline{a, b} = \overline{x_0, y_0} \xrightarrow{1} \overline{x_1, y_1} \xrightarrow{1} \cdots \xrightarrow{1} \overline{x_n, y_n} = \overline{c, d},$$

then we say that a, b is weakly projective to c, d, in notation: $\overline{a, b} \to \overline{c, d}$, or if we are also interested in the number n, then we write $\overline{a, b} \xrightarrow{n} \overline{c, d}$.

If $\overline{a, b} \xrightarrow{1} \overline{c, d}$ and $\overline{c, d} \xrightarrow{1} \overline{a, b}$, then a, b and c, d are transposes, and we write $\overline{a, b} \xleftrightarrow{1} \overline{c, d}$. If the sequence (1) may be chosen in such a way that the neighbouring members are transposes, then a, b and c, d are called projective, and we write $\overline{a, b} \leftrightarrow \overline{c, d}$.

The notion of weak projectivity is due to R. P. Dilworth [8] (see also Malcev [19], Grätzer and Schmidt [12]). Dilworth uses his terminology just reversed as we do.

The importance of this notion is shown by the fact that $\overline{a, b} \to \overline{c, d}$ and $a \equiv b\ (\Theta)$ imply $c \equiv d\ (\Theta)$ (applying this to $\Theta = \omega$, we get that $a = b$ implies $c = d$, a fact which will be used several times).

Now we are able to describe Θ_{ab}:

Theorem I. (R. P. Dilworth [8].) *Let a, b, c, d be elements of the lattice L. $c \equiv d\ (\Theta_{ab})$ holds if and only if there exist $y_i \in L$ with*

$$(2) \qquad c \cup d = y_0 \geqq y_1 \geqq \cdots \geqq y_k = c \cap d \quad \text{and} \quad \overline{a, b} \to \overline{y_{i-1}, y_i} \qquad (i = 1, 2, \ldots, k).$$

It is easy to describe $\Theta[H]$, using Lemma III, Theorem I and the following trivial identity:

$$(3) \qquad \Theta[H] = \bigvee_{a, b \in H} \Theta_{ab}.$$

The symbol $\Theta[H]$ will be used mostly in case H is an ideal. Then one can prove the following important identity (see [14]):

$$(4) \qquad \Theta[\vee I_\alpha] = \vee \Theta[I_\alpha] \qquad (I_\alpha \in I(L)).$$

The following definition is of central importance in this paper. Let L be a lattice and I an ideal of L. By the factor lattice L/I of the lattice L modulo the ideal I is meant the homomorphic image of L induced by $\Theta[I]$, i. e.

$$L/I \simeq L(\Theta[I]).$$

Finally, we mention the definition of permutability: the congruence relations Θ and Φ are called permutable if $a \equiv x$ (Θ) and $x \equiv b$ (Φ) imply the existence of a y such that $a \equiv y$ (Φ) and $y \equiv b$ (Θ).

§ 3. Lattices and elements with special properties

2 will denote the lattice of two elements.

Let U denote the non-modular lattice of five elements, generated by the elements p, q, r, that is, $p > q$, $p \cup r = q \cup r = i$, $p \cap r = o$. V will denote the modular, non-distributive lattice of five elements with the generators p, q, r, that is, $p \cup q = q \cup r = r \cup p = i$, $p \cap q = q \cap r = r \cap p = o$.

An element d of the lattice L is called distributive if

$$d \cup (x \cap y) = (d \cup x) \cap (d \cup y) \tag{5}$$

for all $x, y \in L$. In [25] O. Ore has proved that d is distributive if and only if $x \equiv y$ $(\Theta[(d]])$ implies $x \cup y = [(x \cap y) \cup d] \cap (x \cup y)$.

An element n of L is said to be neutral if the sublattice $\{n, x, y\}$ is distributive, where x and y are arbitrary elements from L. The following theorem will be useful:

Theorem II. (Ore [24].) *The elements $x, y, z \in L$ generate a distributive sublattice of L if and only if for all permutations a, b, c of x, y, z the following equalities hold:*

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c), \tag{6}$$

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c), \tag{7}$$

$$(a \cap b) \cup (b \cap c) \cup (c \cap a) = (a \cup b) \cap (b \cup c) \cap (c \cup a). \tag{8}$$

Corollary. *An element n of L is neutral if and only if for all $x, y \in L$ the five equalities obtained from* (6)—(8) *by substituting permutations of x, y, n hold.*

Remark. It will follow from the theory of standard elements that this corollary may be sharpened, omitting three from the five conditions.

Theorem III. (Birkhoff [5].) *An element n of L is neutral if and only if*

(i) $\quad n \cup (x \cap y) = (n \cup x) \cap (n \cup y)$ *for all* $x, y \in L$;

(i′) $\quad n \cap (x \cup y) = (n \cap x) \cup (n \cap y)$ *for all* $x, y \in L$;

(ii) $\quad n \cap x = n \cap y$ *and* $n \cup x = n \cup y$ $(x, y \in L)$

imply $x = y$, i.e. the relative complements of n are unique.

Theorem IV. (Ore [24].) *An element n of a modular lattice L is neutral if and only if condition* (i) *(or equivalently, condition* (i′)*) is satisfied.*

An ideal I of L is called distributive if it is a distributive element of $I(L)$. I is neutral if it is a neutral element of $I(L)$.

The lattice L is weakly modular (see GRÄTZER and SCHMIDT [12]) if from $a, b \to c, d$ $(a, b, c, d \in L; c \neq d)$ it follows the existence of $a_1, b_1 \in L$ satisfying $a \cap b \leqq a_1 < b_1 \leqq a \cup b$ and $c, d \to \overline{a_1, b_1}$.

LEMMA IV. (GRÄTZER and SCHMIDT [12].) *Let the lattice L be*

A) *modular, or*

B) *relatively complemented, or*

C) *simple.*

Then L is weakly modular.

A lattice L with zero is called section complemented if all of its intervals of type $[0, a]$ are complemented as lattices. In general, the lattice L is section complemented if any element of L is contained in a suitable principal dual ideal which is section complemented as a lattice.[5]

The following assertion is trivial:

LEMMA V. *Any relatively complemented lattice is section complemented.*

Finally, we mention the $\bigvee$-distributive law:

$$x \cap \bigvee y_\alpha = \bigvee (x \cap y_\alpha).$$

A complete lattice L is called $\bigvee$-distributive if this law unrestrictedly holds in L.

Of importance is the theorem of FUNAYAMA and NAKAYAMA that asserts: $\Theta(L)$ is $\bigvee$-distributive.

The partition lattice $P(H)$ of the set H is defined as the partially ordered set of all partitions of H, where the partition p is said to be smaller than q if p is a refinement of q.

CHAPTER II

STANDARD ELEMENTS AND IDEALS

§ 1. Standard elements

We begin with repeating the definition of standard elements:

The element s of the lattice L is standard if the equality

(9) $$x \cap (s \cup y) = (x \cap s) \cup (x \cap y)$$

holds for all $x, y \in L$.

[5] The section complemented lattices with zero are called by HERMES [16] „abschnittkomplementäre Verbände". The English name was suggested by Mr. LORENZ.

First of all, let us see some examples for standard elements. In the lat ice U (see Chapter I, § 3) p is a standard element. At the same time, it is clear that p is not neutral. (Furthermore, in the same lattice $(r]$ is a homomorphism kernel, but r is not standard.)

Obviously, any element of a distributive lattice is standard. Furthermore, in any lattice the elements 0 and 1 (if exist) are standard elements.

The simplest form for defining standard elements is the equality (9), however, it is not the most important property of a standard element. Some important characterizations of standard elements are given in

THEOREM 1. (The fundamental characterization theorem of standard elements.) *The following conditions upon an element s of the lattice L are equivalent:*

(α) *s is a standard element;*

(β) *the equality $u=(u\cap s)\cup(u\cap t)$ holds whenever $u\leq s\cup t$ $(u, t\in L)$;*

(γ) *the relation Θ_s, defined by "$x\equiv y$ (Θ_s) if and only if $(x\cap y)\cup s_1=x\cup y$ for some $s_1\leqq s$", is a congruence relation;*

(δ) *for all $x, y\in L$*

(i) $$s\cup(x\cap y)=(s\cup x)\cap(s\cup y),$$

(ii) $$s\cap x=s\cap y \quad \textit{and} \quad s\cup x=s\cup y \quad \textit{imply} \quad x=y.$$

PROOF. We will prove the equivalence of the four conditions cyclically

(α) *implies* (β). Indeed, if (α) holds and $u\leq s\cup t$, then $u=u\cap(s\cup t)$. Owing to (9) we get $u=(u\cap s)\cup(u\cap t)$, which was to be proved.

(β) *implies* (γ). Using condition (β) and Lemma II we will prove that Θ_s as defined above is a congruence relation.

(a) $x\equiv x$ (Θ_s). Indeed, for any $x\in L$, the equality $(x\cap x)\cup(x\cap s)=x$ trivially holds, so if we put $s_1=x\cap s$, we get the assertion.

(b) $x\cap y\equiv x\cup y$ (Θ_s) if and only if $x\equiv y$ (Θ_s). This is trivial from the definition of Θ_s.

(c) $x\geq y\geq z$, $x\equiv y$ (Θ_s) and $y\equiv z$ (Θ_s) imply $x\equiv z$ (Θ_s). By hypothesis $x=y\cup s_1$ and $y=z\cup s_2$ for suitable elements $s_1, s_2\leqq s$. Consequently, $x=y\cup s_1=(z\cup s_2)\cup s_1=z\cup(s_1\cup s_2)$ for $s_1\cup s_2\leqq s$, that means, $x\equiv z$ (Θ_s).

(d) In case $x\geq y$ and $x\equiv y$ (Θ_s) hold, $x\cup z\equiv y\cup z$ (Θ_s) and $x\cap z\equiv y\cap z$ (Θ_s). In fact, by assumption $x=y\cup s_1$ $(s_1\leq s)$, and hence we get $x\cup z=(y\cup z)\cup s_1$, that is $x\cup z\equiv y\cup z$ (Θ_s). To prove the second assertion, we start from the relations $x=y\cup s_1$ and $x\cap z\leqq y\cup s_1\leqq y\cup s$. Applying condition ($\beta$) to $u=x\cap z, t=y$ and using $x\cap y=y$, we get

$$x\cap z=(x\cap z\cap s)\cup(x\cap z\cap y)=(y\cap z)\cup s_2,$$

where $s_2=x\cap z\cap s\leqq s$, which means $x\cap z\equiv y\cap z$ (Θ_s).

(γ) *implies* (δ). First we prove that (γ) implies (i). According to the definition of Θ_s, the congruences $x \equiv s \cup x$ (Θ_s) and $y \equiv s \cup y$ (Θ_s) hold for arbitrary $x, y \in L$. We get $x \cap y \equiv (s \cup x) \cap (s \cup y)$ (Θ_s). By monotonicity, $x \cap y \leqq (s \cup x) \cap (s \cup y)$, hence again by the definition of Θ_s it follows that $(s \cup x) \cap (s \cup y) = (x \cap y) \cup s_1$ with suitable $s_1 \leqq s$. Joining with s and keeping the inequalities $s_1 \leqq s$ and $s \leqq (s \cup x) \cap (s \cup y)$ in view, we derive $s \cup (x \cap y) = (s \cup x) \cap (s \cup y)$, which is nothing else than (i).

Secondly, we prove that (γ) implies (ii). Let the elements x and y be chosen as in (ii). We know that $s \cup y \equiv y$ (Θ_s), so meeting with x and using $x \cup s = y \cup s$ we get $x = (x \cup s) \cap x = (y \cup s) \cap x \equiv y \cap x$ (Θ_s), consequently, using (γ), $(x \cap y) \cup s_1 = x$ with suitable $s_1 \leqq s$. From the last equality $s_1 \leqq x$, accordingly, $s_1 \leqq s \cap x = s \cap y \leqq y$ (in the meantime we have used the supposition $s \cap x = s \cap y$ of (ii)), thus $x = (x \cap y) \cup s_1 \leqq (x \cap y) \cup y = y$. We may conclude similarly that $y \leqq x$, and thus $x = y$, which was to be proved.

(δ) *implies* (α). Let x and y be arbitrary elements of L and define $a = x \cap (s \cup y)$ and $b = (x \cap s) \cup (x \cap y)$. By (ii), it suffices to prove that $s \cap a = s \cap b$ and $s \cup a = s \cup b$.

To prove the first equality we start from $s \cap a$:

$$s \cap a = s \cap [x \cap (s \cup y)] = x \cap [s \cap (s \cup y)] = x \cap s.$$

It follows from the monotonicity that $x \cap s \leqq b = (x \cap s) \cup (x \cap y) \leqq [x \cap (s \cup y)] \cup [x \cap (s \cup y)] = a$. Meeting with s, we get $s \cap x \leqq s \cap b \leqq s \cap a$. But we have already proved that $s \cap x = s \cap a$, and so $s \cap a = s \cap b$. To prove $s \cup a = s \cup b$ we start from $s \cup a$ and use (i) several times:

$$s \cup a = s \cup [x \cap (s \cup y)] = (s \cup x) \cap [s \cup (s \cup y)] = (s \cup x) \cap (s \cup y) =$$
$$= s \cup (x \cap y) = s \cup (x \cap s) \cup (x \cap y) = s \cup b,$$

and so Theorem 1 is completely proved.

Rewriting (i) and weakening (ii), (δ) may be transformed to the following form:

LEMMA 1. *An element s of L is standard if and only if the following two conditions are satisfied:*

(i*) *the correspondence $x \to x \cup s$ is an endomorphism of L;*

(ii*) *if $x \geqq y$, $s \cup x = s \cup y$ and $s \cap x = s \cap y$, then $x = y$.*

It is easy to see that (i) is equivalent to (i*). Indeed, for any fixed s, the correspondence $x \to x \cup s$ is a join-endomorphism. That it is meet-endomorphism as well, is guaranteed just by (i). In the proof of Theorem I, at the step "(δ) implies (α)" we have used (ii) only for $x = a$ and $y = b$, and

in this case $y \leq x$ holds. Consequently, in the proof we have only used (ii*), and so one can replace (ii) by (ii*).

From condition (γ) of Theorem 1 we derive easily:

LEMMA 2. *Let s be a standard element of the lattice L. Then $(s]$ is a homomorphism kernel, namely $\Theta[(s]] = \Theta_s$. Conversely, if $x \equiv y$ $(\Theta[(s]])$ holds when and only when $(x \cap y) \cup s_1 = x \cup y$ with a suitable $s_1 \leqq s$, then s is a standard element.*

PROOF. The congruence relation Θ_s obviously satisfies $\Theta_s = \Theta[(s]]$, consequently $(s]$ is in the kernel of the homomorphism induced by Θ_s. We have to prove that $(s]$ is just the kernel. Otherwise there exists an $x > s$ with $x \equiv s$ (Θ_s). By definition, it follows $x = s \cup s_1$ $(s_1 \leqq s)$ which is obviously a contradiction. Conversely, if $\Theta[(s]] = \Theta_s$, then Θ_s is a congruence relation, since $\Theta[(s]]$ is one, and then from condition (γ) of Theorem 1 it follows that s is a standard element.

We have formulated Lemma 2 separately — despite the fact that it is an almost trivial variant of condition (γ) of Theorem 1 — because it points out that property of the standard elements which we think to be the most important one. It may be reformulated as follows: if $(s]$ is a principal ideal of L, then $x \equiv y$ $(\Theta[(s]])$ if and only if there exist a sequence of elements $x \cup y = z_0 \geqq z_1 \geqq \cdots \geqq z_m = x \cap y$ of L, an $s_1 \leqq s$, and a sequence of integers $n_1, \ldots, n_m$ such that $\overline{s_1, s} \xrightarrow{n_i} \overline{z_{i-1}, z_i}$ $(i = 1, \ldots, m)$. Now the definition of standardness is as follows: s is standard if and only if $n_i = 1$ may be chosen for all i. It follows then we may suppose $m = 1$ as well.

§ 2. Standard ideals

An ideal S of the lattice L is called standard if it is a standard element of the lattice $I(L)$, that is, if

$$I \cap (S \cup K) = (I \cap S) \cup (I \cap K) \tag{10}$$

holds for any pair of ideals I, K of L.

An example for standard ideals is given by the ideal $(p]$ of the lattice U. Further examples will be given at the end of this section.

Our chief aim in this section is to prove the analogue of Theorem 1 for standard ideals.

THEOREM 2. (The fundamental characterization theorem of standard ideals.) *The following seven conditions for an ideal S of the lattice L are equivalent:*

(α') *S is a standard ideal;*

(α'') *the equality*

$$I \cap (S \cup K) = (I \cap S) \cup (I \cap K)$$

holds if I and K are principal ideals;

(β') *for any ideal I, the elements of $S \cup I$ are of the form $s \cup x$ ($s \in S$, $x \in I$);*

(β'') *for any principal ideal I, the elements of $S \cup I$ are of the form $s \cup x$ ($s \in S$, $x \in I$);*

(γ') *the relation Θ_S of $I(L)$ defined by "$I \equiv K$ (Θ_S) if and only if $(I \cap K) \cup S_1 = I \cup K$ with a suitable $S_1 \subset S$" is a congruence relation of $I(L)$;*

(γ'') *the relation $\Theta[S]$ of L defined by "$x \equiv y$ ($\Theta[S]$) if and only if $(x \cap y) \cup s = x \cup y$ with a suitable $s \in S$" is a congruence relation;*

(δ') *for all I and $K \in I(L)$*

(i) $$S \cup (I \cap K) = (S \cup I) \cap (S \cup K),$$

(ii) $S \cap I = S \cap K$ *and* $S \cup I = S \cup K$ *imply* $I = K$.

PROOF. The conditions of this theorem are the analogues of those of Theorem 1. To make the similarity clear, first we show that (β') is equivalent to the following condition:

(β^*) if for the ideals I and J the inequality $J \subset S \cup I$ holds, then $J = (J \cap S) \cup (J \cap I)$.

It is, obviously, equivalent to (β^*) that any element of J may be written in the form $s \cup x (s \in S, x \in I)$. Since J is arbitrary, that means: any element of $S \cup I$ is of the form $s \cup x$, and this is condition (β'). So these two conditions are equivalent.

Another analogue of (β'') may also be formulated:

(β^{**}) if for the principal ideals I and J the inequality $J \subseteq S \cup I$ holds, then $J = (J \cap S) \cup (J \cap I)$.

Now, it is trivial that the equivalence of (α'), (β'), (γ') and (δ') is an immediate consequence of Theorem 1.

(α'') is a special case of (α'). The proofs that (α'') implies (β'') and (β'') implies (γ'') run on the similar lines as those of the corresponding implications in the proof of Theorem 1. Thus it is enough to prove that (γ'') implies (β'). Suppose (γ'') holds and let I be an arbitrary ideal of L, and $x \in S \cup I$. From Lemma I we get the existence of $s \in S$ and $i \in I$ with $x \leqq s \cup i$. Since $s \equiv s \cap i$ ($\Theta[S]$), therefore $s \cup i \equiv (s \cap i) \cup i = i$ ($\Theta[S]$), and so $x = x \cap (s \cup i) \equiv x \cap i$ ($\Theta[S]$). Accordingly, using (γ'') we get $x = (x \cap i) \cup s'$ where $s' \in S$. But $x \cap i \in I$, hence (β') is proved.

The proof of Theorem 2 is complete.

The analogues of Lemmas 1 and 2 are naturally true. We formulate only the analogue of the most important part of Lemma 2.

LEMMA 3. *Let S be a standard ideal of L. Then the congruence relation $\Theta[S]$ of L defined by condition (γ'') of Theorem 2 is the congruence relation induced by S and S is the kernel of the homomorphism induced by $\Theta[S]$.*

We may say that Lemma 3 gives an approval of the notation we have used in condition (γ'') of Theorem 2.

We get many examples of standard ideals from the following

LEMMA 4. *The principal ideal $(s]$ of L is standard if and only if s is a standard element of L.*

PROOF. The assertion is clear comparing Lemma 2 with condition (γ'') of Theorem 2, since $s_1 \in (s]$ and $s_1 \leqq s$ are equivalent statements.

It follows now from Lemma 4 that the existence of standard elements in a lattice implies the existence of standard ideals. The converse of this statement is not true. We construct a lattice L in which there exists a standard ideal, but has no standard element. Consider the direct product of the chain of the integers with **2**. The elements of this lattice are of the form $(n, 0)$ and $(n, 1)$ where n is an arbitrary integer and 0 and 1 are the elements of **2**. We define new elements x_n $(n = 0, \pm 1, \ldots)$, subject to the following relations:

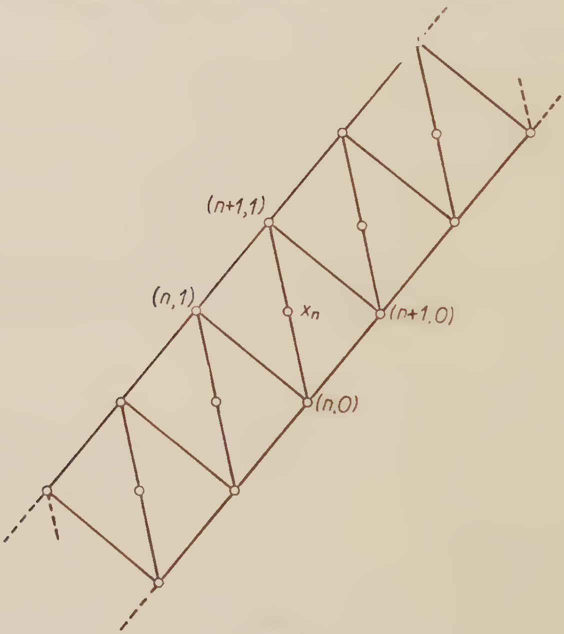

Fig. 2

$$x_n \cup (n, 1) = x_n \cup (n+1, 0) = (n+1, 1),$$
$$x_n \cap (n, 1) = x_n \cap (n+1, 0) = (n, 0).$$

The resulting partially ordered set L is shown in Fig. 2. One can easily

prove that L is a lattice and L is simple, that is, $\Theta(L)$ consists of two elements. In L there is no standard element (if s were one, then $\omega < \Theta_s < \iota$ would be a contradiction), but the whole lattice is a standard ideal.

A proper standard ideal is obtained if we take two copies of this lattice, L_1 and L_2, and define $L \sim L_1 \oplus L_2$. Then this lattice contains no standard element, but L_1 and L_2 are standard ideals.

It is natural to ask, why the following condition is not included in Theorem 2:

(δ'') if I and K are principal ideals, then

(i) $$S \cup (I \cap K) = (S \cup I) \cap (S \cup K);$$

(ii) $$S \cap I = S \cap K \quad \text{and} \quad S \cup I = S \cup K \text{ imply } I = K.$$

The reason is that we could not prove the equivalence of this condition to the others. Therefore we ask

PROBLEM 1. Does condition (δ'') characterize the standardness of the ideal S?

§ 3. Basic properties of standard elements and ideals

In this section and in the next one we shall deduce from the fundamental characterization theorems some important properties of standard elements and ideals.

If S is a standard ideal, then we call the congruence relation $\Theta[S]$ generated by S a standard congruence relation. If $S = (s]$, then $\Theta[S] = \Theta_s$, so Θ_s is a standard congruence relation that we may call principal standard congruence relation. First we see some results on the connection between standard ideals and standard congruence relations.

THEOREM 3. *The standard elements form a distributive sublattice of the lattice L. The principal standard congruence relations form a sublattice of $\Theta(L)$. Between these two lattices the correspondence $s \to \Theta_s$ is an isomorphism.*

Further, the standard ideals form a $\vee$-distributive sublattice of $I(L)$ which is closed under forming complete join. The standard congruence relations form a sublattice of $\Theta(L)$. The correspondence $S \to \Theta[S]$ is an isomorphism between these two lattices.

PROOF. First we verify the assertions concerning standard elements. Let s_1 and s_2 be standard. Then by an iterated use of (9) we get that for all $x, y \in L$

$$x \cap [(s_1 \cup s_2) \cup y] = x \cap [s_1 \cup (s_2 \cup y)] = (x \cap s_1) \cup [x \cap (s_2 \cup y)] =$$
$$= (x \cap s_1) \cup (x \cap s_2) \cup (x \cap y) = [x \cap (s_1 \cup s_2)] \cup (x \cap y),$$

that means, by definition, that $s_1 \cup s_2$ is standard. It is almost trivial that the correspondence $s \to \Theta_s$ is a join-endomorphism. Indeed, owing to Lemma 2 and the standardness of $s_1 \cup s_2$, the equality $\Theta_{s_1} \cup \Theta_{s_2} = \Theta_{s_1 \cup s_2}$ is equivalent to $\Theta[(s_1]] \cup \Theta[(s_2]] = \Theta[(s_1 \cup s_2]]$, and this is a special case of formula (4). Further, if $s_1 \neq s_2$, then $\Theta_{s_1} \neq \Theta_{s_2}$, for the kernels of the homomorphisms induced by Θ_{s_1}, resp. Θ_{s_2} are different (see Lemma 2).

Now we prove $\Theta_{s_1} \cap \Theta_{s_2} = \Theta_{s_1 \cap s_2}$. If $x \equiv y$ $(\Theta_{s_1} \cap \Theta_{s_2})$, then $x \equiv y$ (Θ_{s_1}), and so $(x \cap y) \cup s_1' = x \cup y$ $(s_1' \leqq s_1)$, on the other hand $x \equiv y$ (Θ_{s_2}) holds as well, and from this $s_1' = (x \cup y) \cap s_1' \equiv (x \cap y) \cap s_1'$ (Θ_{s_2}), hence with a suitable $s \leqq s_2$ the relation $s_1' = [(x \cap y) \cap s_1'] \cup s$ holds. Consequently, $s \leqq s_1'$ is valid, therefore $s \leqq s_1 \cap s_2$ and $(x \cap y) \cup s = (x \cap y) \cup [(x \cap y) \cap s_1'] \cup s = x \cup y$. We have proved the following: $x \equiv y$ $(\Theta_{s_1} \cap \Theta_{s_2})$ if and only if $(x \cap y) \cup s = x \cup y$ with a suitable $s \in (s_1 \cap s_2]$. According to Lemma 2, this means that $s_1 \cap s_2$ is standard and $\Theta_{s_1} \cap \Theta_{s_2} = \Theta_{s_1 \cap s_2}$. Thus we have shown that the standard elements form a sublattice of L, the principal standard congruence relations form a sublattice of $\Theta(L)$, and the correspondence $s \to \Theta_s$ is an isomorphism. It follows now, since $\Theta(L)$ is a distributive lattice, that the lattice of standard elements is distributive.

Applying the results proved so far to the lattice of all ideals of L, we get that the standard ideals form a sublattice of $I(L)$, the congruence relations Θ_S form a sublattice of $\Theta(I(L))$, and $S \to \Theta_S$ is an isomorphism. But we need the same assertions for $\Theta[S]$ instead of Θ_S. Therefore we prove a lemma from which the desired conclusion will follow.

First we need some notions. Let Θ be a congruence relation of L; Θ defines in the natural way a homomorphism of $I(L)$ under which $I \equiv J$ $(I, J \in I(L))$ if and only if to any $x \in I$ there exists a $y \in J$ such that $x \equiv y$ (Θ), and conversely. That means: $I \equiv J$ if and only if the images of I and J under the homomorphism $L \to L(\Theta)$ are the same. This congruence relation of $I(L)$ we call the extension of Θ to $I(L)$. On the other hand, any congruence relation Φ of $I(L)$ induces a congruence relation of L under which $x \equiv y$ if and only if $(x] \equiv (y]$ (Φ). This we call the restriction of Φ to L. Now we may state

Lemma 5. *Let S be a standard ideal. Then Θ_S is the extension of $\Theta[S]$ to $I(L)$ and $\Theta[S]$ is the restriction of Θ_S to L.*

Proof. Let $\bar{\Theta}[S]$ be the extension of $\Theta[S]$ to $I(L)$ and $I \equiv J$ $(\bar{\Theta}[S])$; we suppose $I \subseteq J$. Choosing a $y \in J$ we can find an $x \in I$ $(y \geqq x)$ with $x \equiv y$ $(\Theta[S])$, and so there exists an s_{xy} with $x \cup s_{xy} = y$. The ideal S' generated by the s_{xy} (x and y run over the elements of I and J) satisfies $S' \subseteq S$ and $I \cup S' = J$. On the other hand, if $I \cup S' = J$ with a suitable $S' \subseteq S$,

then with $y \in J$ it follows that $y = s \cup x$ ($s \in S'$, $x \in I$, S is standard!), and so $x \equiv y$ ($\Theta[S]$). Thus $\bar{\Theta}[S] = \Theta_S$. To show the second assertion, suppose $(a] \equiv (b]$ (Θ_S). Then there exists an $S' \subseteq S$ with $(a \cap b] \cup S' = (a \cup b]$. It follows that $a \cup b \in (a \cap b] \cup S$, and since S is standard, we may find an $s \in S$ with $(a \cap b) \cup s = a \cup b$, which proves $a \equiv b$ ($\Theta[S]$).

COROLLARY. *The correspondence $\Theta[S] \to \Theta_S$ is an isomorphism between the lattice of all standard congruence relations of L and the lattice of all principal standard congruence relations of $I(L)$.*

Combining Corollary of Lemma 5 with the facts we have proved above, we get all the assertions of Theorem 3 with the exception of the statement that the lattice of all standard ideals of L is closed under forming complete join and is $\bigvee$-distributive.

Suppose the S_α are standard ideals, $S = \bigvee S_\alpha$, I is an arbitrary ideal and $x \in I \cup S$. Owing to Lemma 1 we may find $s_i \in S_{\alpha_i}$, $y \in I$ such that $x \leq \bigvee_{i=1}^{n} s_i \cup y$, consequently, $x \in \bigvee_{i=1}^{n} S_{\alpha_i} \cup I$. We know already that $\bigvee_{i=1}^{n} S_{\alpha_i}$ is a standard ideal, hence $x = u \cup r$, $u \in \bigvee_{i=1}^{n} S_{\alpha_i} \subseteq S$ and $r \in I$. Thus, by condition (β') of Theorem 2, S is a standard ideal.

Now we may apply formula (4) which compared with Lemma 3 gives $\Theta[\bigvee S_\alpha] = \bigvee \Theta[S_\alpha]$. Thus the standard congruence relations form a sublattice of $\Theta(L)$ which is closed under forming complete join. In $\Theta(L)$ there holds the $\bigvee$-distributive law and this is preserved under taking a sublattice which is closed under forming complete join, therefore the lattice of standard congruence relations is $\bigvee$-distributive, and then the same is true for the lattice of standard ideals. Thus the proof of Theorem 3 is completed.

Naturally arises the question: is the complete meet of standard ideals (if it exists) a standard ideal? We will show by a counterexample that this is not true in general. Let N be the chain of all negative integers, and denote by $0, a, b, 1$ the elements of a Boolean algebra of four elements and

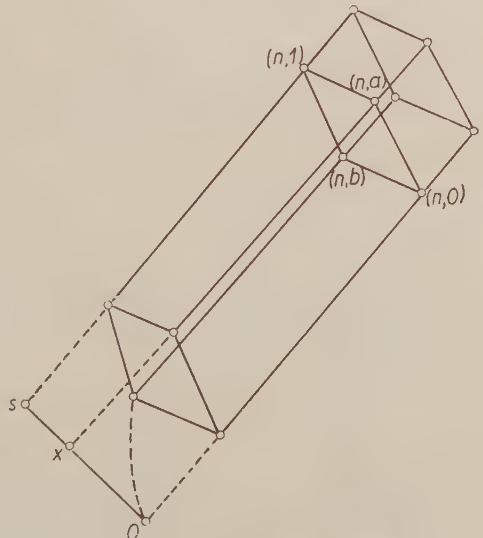

Fig. 3

form the direct product of N and this Boolean algebra. The lattice L is formed by adding three further elements $s, x, 0$ to this direct product subject to

$$\left.\begin{aligned} &x \cap (n, b) = 0, \quad x \cup (n, b) = (n, 1), \quad x \cup (n, a) = (n, a), \\ &s \cup (n, a) = s \cup (n, b) = (n, 1), \quad s \cap (n, a) = x, \quad s \cap (n, b) = 0 \end{aligned}\right\} \quad (n \in N).$$

The resulting partially ordered set is given by Fig. 3. It is easy to prove that L is a lattice. Define $s_i = (-i, 1)$ $(i = 1, 2, \ldots)$. The principal ideals $(s_i]$ are standard, while their complete meet $(s]$ is not a standard ideal, for

$$(n, b) \cap [s \cup (n, a)] = (n, b),$$
$$[(n, b) \cap s] \cup [(n, b) \cap (n, a)] = 0 \cup (n, 0) = (n, 0),$$

and so (9) does not hold.

Owing to the definition, the following assertion is immediate:

Lemma 6. *Let the correspondence $x \to \bar{x}$ be the homomorphism of a lattice L onto a lattice $\bar{L}$. If s is a standard element of L, then $\bar{s}$ is a standard element of $\bar{L}$.*

Corollary. *Let $x \to \bar{x}$ be a homomorphism of L onto $\bar{L}$, let S be an ideal of L, and denote by $\bar{S}$ the homomorphic image of S under this homomorphism. If S is standard in L, then $\bar{S}$ is standard in $\bar{L}$.*

Proof of the Corollary. Let Θ be the congruence relation which induces the homomorphism $x \to \bar{x}$. Then the extension $\bar{\Theta}$ of Θ to $I(L)$ (defined before Lemma 5) is a congruence relation of $I(L)$ and $(x] \equiv (y]$ $(\bar{\Theta})$ $(x, y \in L)$ if and only if $x \equiv y$ (Θ). Hence the homomorphism $X \to \bar{X}$ $(X \in I(L))$ induced by $\bar{\Theta}$ is an extension of the homomorphism $x \to \bar{x}$ and carries S onto $\bar{S}$. Thus we may apply Lemma 6 to $I(L)$ and get the Corollary.

The converse of Lemma 6 is not true. One can find easily a lattice L, a homomorphism $x \to \bar{x}$ of L onto $\bar{L}$ and in L a standard element $\bar{s}$ such that in L there is no standard element s with $s \to \bar{s}$. As an example take the lattice U (see § 3 of Chapter I) and the homomorphism induced by Θ_{pq}. In the homomorphic image of U (which is the Boolean algebra of four elements) the image of r is standard, while r is not standard and is not congruent to any standard element (as a matter of fact, r forms alone a congruence class under Θ_{pq}).

From the point of view of later applications it is of importance the

Lemma 7. *Any two standard congruence relations are permutable.*

Proof. We have to prove that if S and T are standard ideals, x, y and z elements of the lattice with $x \equiv y$ $(\Theta[S])$, $y \equiv z$ $(\Theta[T])$, then for a suitable element u the relations $x \equiv u$ $(\Theta[T])$, $u \equiv z$ $(\Theta[S])$ hold.

First we consider the case $x \geqq y \geqq z$. Then by condition (γ'') of Theorem 2, we get elements $s \in S$ and $t \in T$ with $x = y \cup s$, $y = z \cup t$. We assert that $u = z \cup s$ fulfils the requirements. Indeed, $z \equiv z \cup s = u$ $(\Theta[S])$ and because of $u \cup t = z \cup s \cup t = y \cup s = x$ we have $u \equiv u \cup t = x$ $(\Theta[T])$.

In the general case consider the elements $x, x \cup y, x \cup y \cup z$. We have $x \equiv x \cup y$ $(\Theta[S])$ and $x \cup y \equiv x \cup y \cup z$ $(\Theta[T])$, therefore with a suitable element v, $x \equiv v$ $(\Theta[T])$ and $v \equiv x \cup y \cup z$ $(\Theta[S])$. We obtain in a similar way the existence of a w with $z \equiv w$ $(\Theta[S])$ and $w \equiv x \cup y \cup z$ $(\Theta[T])$. The element $u = v \cap w$ fulfils the requirements, for $u = v \cap w \equiv v \cap (x \cap y \cap z) = v$ $(\Theta[T])$ and this, together with $v \equiv x$ $(\Theta[T])$, gives $u \equiv x$ $(\Theta[T])$. Similarly, we can prove $u \equiv z$ $(\Theta[S])$, completing the proof of the lemma.

Let s be a standard element of the lattice L. Then from Lemma 2 it is clear that $L\,(s] \simeq [s)$. Indeed, for all $x \in L$ we have $x \equiv s \cup x$ (Θ_s), and so any element of L is congruent to a suitable element of $[s)$, therefore $L/(s]$ is a homomorphic image of $[s)$. But this homomorphism is an isomorphism, for $x \geqq y \geqq s$ and $x \equiv y$ (Θ_s) implies $x \leqq y \cup s = y$, i. e. $x = y$.

To determine the factor lattice modulo a non-principal standard ideal is not so simple A solution of this problem is given in

Theorem 4. *Let S be a standard ideal of the lattice L. Then the lattice of all ideals of L/S is isomorphic to the interval $[S, L]$ of $I(L)$, and, consequently, the interval $[S, L]$ of $I(L)$ determines L/S up to isomorphism.*

Proof. We know from Lemma 5 that the extension of $\Theta[S]$ to $I(L)$ is Θ_S. Hence the homomorphism $L \to L/S$ induces in $I(L)$ a homomorphism $I(L) \to I(L)/(S]$. But, as we have remarked above, $I(L)/(S] \sim [S, L]$ where the interval $[S, L]$ is taken in $I(L)$. This, together with a theorem of Komatu [17], according to which every lattice is determined up to isomorphism by the lattice of its ideals, we get the theorem.

§ 4. Additional properties of standard elements and ideals

In our paper [10] there is a lemma that states: in a distributive lattice if the join and meet of two ideals are principal ideals, then the two ideals themselves are principal. We now generalize this to standard ideals of arbitrary lattices:

Lemma 8. *Let I be an arbitrary and S a standard ideal of the lattice L. If $I \cup S$ and $I \cap S$ are principal, then I itself is principal.*

Proof. Let $I \cup S = (a]$ and $I \cap S = (b]$. By condition (β') of Theorem 2 we have $a = s \cup x$ $(s \in S, x \in I)$. We state that $I = (x \cup b]$. Indeed, suppose

$w \geqq x \cup b$ and $w \in I$. Then $(a] \supseteq S \cup (w] \supseteq S \cup (x \cup b] \supseteq S \cup (x] = (a]$, that is, $S \cup (a] = S \cup (x \cup b]$. Further, $(b] = S \cap I \supseteq S \cap (w] \supseteq S \cap (x \cup b] \supseteq S \cap (b] = (b]$, and so $S \cap (w] = S \cap (x \cup b]$. This two equalities imply (see condition (ii) of (δ') of Theorem 2) that $(w] = (x \cup b]$, and so $w = x \cup b$. Therefore, there are no elements in I greater than $x \cup b$, that is, $I = (x \cup b]$, completing the proof of the lemma.

By means of a simple example one can show that under the hypothesis of Lemma 8 S is not necessarily a principal ideal.

Since the ideals of a distributive lattice are standard, an exact analogue of the lemma of [10] is the

Corollary. *If the join and meet of two standard ideals are principal, then both standard ideals are principal.*

This corollary does not call for proof.

Lemma 9. *Let s be a standard element of the lattice L and a an arbitrary element of L. Then $a \cap s$ is a standard element of the lattice $(a]$.*

Proof. Any element of the ideal $(a]$ may be written in the form $a \cap x$ $(x \in L)$. Hence it is enough to prove that

$$(x \cap a) \cap [(s \cap a) \cup (y \cap a)] = [(x \cap a) \cap (s \cap a)] \cup [(x \cap a) \cap (y \cap a)].$$

Starting from the left member and applying (9) repeatedly, we get

$$(x \cap a) \cap [(s \cap a) \cup (y \cap a)] = (x \cap a) \cap [(s \cup y) \cap a] = (x \cap a) \cap (s \cup y) =$$
$$= (x \cap a \cap s) \cup (x \cap a \cap y) = [(x \cap a) \cap (s \cap a)] \cup [(x \cap a) \cap (y \cap a)]$$

which was to be proved.

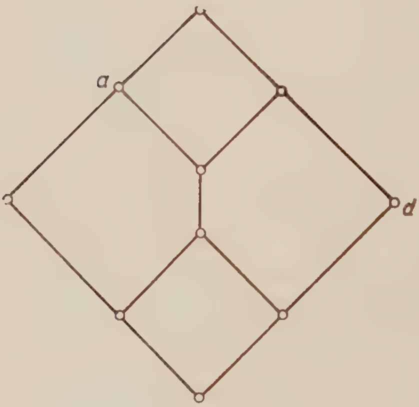

Fig. 4

Corollary. *Let S be a standard ideal and I an arbitrary ideal of the lattice L. Then $S \cap I$ is a standard ideal of the lattice I.*

Perhaps it is not worthless to note that the conclusions of this lemma are not valid for distributive elements. A counterexample is the lattice of Fig. 4, where d is a distributive element of the lattice, but the element $a \cap d$ is not a distributive element of the lattice $(a]$.

As we have seen, the neutrality of the element n was defined in such a way that for all $x, y \in L$ the sublattice $\{n, x, y\}$ is distributive. Though, in general, the notion of standard elements does not coincide with the notion of ne-

utral elements, we may hope the validity of a weaker assertion for standard elements. Indeed, the following analogue of the definition of neutral elements is true:

THEOREM 5. *Let s_1 and s_2 be standard elements of the lattice L. Then the sublattice $\{s_1, s_2, x\}$ of L is distributive for all $x \in L$.*

PROOF. Our proof is based upon Theorem II. According to this, we have to prove the validity of (6), (7) and (8).

Condition (7) is valid, for it asserts the same as (9) since b or c is standard. As a consequence of condition (i) of (δ) of Theorem 1, (6) holds if a is standard; otherwise b and c are standard. In this case let us start with the right member of (6), apply (9) for the elements a, $a \cup c$ for the standard element b and then for a, b and the standard element c. We get

$$(a \cup c) \cap (a \cup b) = [(a \cup c) \cap a] \cup [(a \cup c) \cap b] = a \cup (a \cap b) \cup (c \cap b) = a \cup (c \cap b).$$

Finally, we prove (8). (8) is a symmetric function of its variables, therefore we have to prove it for one permutation of its variables only. Using the assertion of Theorem 3, according to which $s_1 \cup s_2$ and $s_1 \cap s_2$ are standard, further equality (9) and condition (i) of (δ) of Theorem 1, we get

$$(s_1 \cap s_2) \cup (s_1 \cap x) \cup (s_2 \cap x) = (s_1 \cap s_2) \cup [(s_1 \cup s_2) \cap x] =$$
$$= [(s_1 \cap s_2) \cup (s_1 \cup s_2)] \cap [(s_1 \cap s_2) \cup x] = (s_1 \cup s_2) \cap [(s_1 \cap s_2) \cup x] =$$
$$= (s_1 \cup s_2) \cap (s_1 \cup x) \cap (s_2 \cup x),$$

and this is just (8). Thus the proof of Theorem 5 is completed.

Applying Theorem 5 to $I(L)$ we get:

COROLLARY. *Let S_1 and S_2 be the standard ideals of the lattice L. Then S_1, S_2 and an arbitrary ideal X of L generate a distributive sublattice of $I(L)$.*

We have got Theorem 5 as an analogue of the definition of neutral elements (ideals). It is natural to ask, whether or not it is possible to get from Theorem 5 a new characterization of standard elements (ideals). That should mean that in a lattice the standard elements form a (unique) maximal subset for which the assertion of Theorem 5 is true. This is not true in general; not even in modular lattices. Consider the lattice V (see § 3 of Chapter I); there are in V only two standard elements: o and i. We may enlarge the set $\{o, i\}$ by the element p, and the assertion of Theorem 5 is true for this enlarged set as well. (See Problem 2.)

Now consider the lattice L and let us fix an element s of L. We call the mapping $x \to (x \cap s, x \cup s)$ of L into $L_s = (s] \times [s)$ the natural mapping of L into L_s. We can obtain by means of this notion a new characterization of

standard elements, which is a direct generalization of a theorem of G. Birkhoff [5].

Theorem 6. *The natural mapping of the lattice L into the lattice L_s is a meet isomorphism if and only if s is a standard element. It is an isomorphism if and only if s is neutral.*

Proof. The first part of this theorem is essentially condition (δ) of Theorem 1. Namely, condition (i) assures that the mapping is a meet-homomorphism and (ii) that different elements have different images. The second part of the theorem is equivalent to Theorem III.

We remark that the idea of the proof of this theorem is due to Birkhoff [5]. This theorem seems to be a good tool for proving the standardness of an element.

Corollary. *Let L be a bounded lattice. $L_s \simeq L$ in the natural way (the direct components are supposed to be ideals of L and $(x, y) \rightarrow x \cup y$) if and only if s is a neutral element having a complement, that is s is an element of the center. In other words, L has a non-trivial direct decomposition if and only if its center has an element different from* 0 *and* 1.

In connection with Theorem 5 the following problem arises:

Problem 2. Is it possible to characterize the set of standard elements as a maximal subset of the lattice L satisfying the condition of Theorem 5 and having some additional properties?

CHAPTER III

STANDARD AND NEUTRAL ELEMENTS

§ 1. Relations between standard and neutral elements

We have mentioned in the Introduction some causes which required the definition of standardness to be a generalization of neutrality and to coincide with the same in modular lattices. It is obvious that our definition fulfils this requirement, that is, the following assertions are valid:

Lemma 10. *All the neutral elements of the lattice L are standard. Furthermore, in modular lattices the two notions coincide. A standard element s is neutral if and only if condition* (i′) *of Theorem III holds.*

Proof. The first assertion is clear from the definitions. The second assertion is a consequence of Theorem IV and we get the third statement

by a simple comparison of the conditions of Theorem 1 (δ) with those of Theorem III.

From these simple observations one can get a result of some interest for neutral ideals.

LEMMA 11. *Let n be a neutral element of the lattice L. Then $(n]$ is a neutral ideal of L and conversely.*

PROOF. Every neutral element is standard, and so by Lemma 4, $(n]$ is standard. Then by Lemma 10 it is enough to prove that

$$(n]\cap(I\cup J)=((n]\cap I)\cup((n]\cap J) \qquad (I, J\in I(L)).$$

Let $A=(n]\cap(I\cup J)$ and $B=((n]\cap I)\cup((n]\cap J)$. Since $A\supseteq B$ holds always, it is enough to prove that $a\in A$ implies $a\in B$. For some $i\in I$ and $j\in J$ we have $a\leqq i\cup j$, and so $a\leqq a\cap(i\cup j)\leqq n\cap(i\cup j)$. But $n\cap(i\cup j)=(n\cap i)\cup(n\cap j)$, because n is neutral and $(n\cap i)\cup(n\cap j)$ is an element of B, hence $a\in B$ holds as well. This completes the proof of the first part of the lemma. The converse statement is trivial.

It is of some interest that we could not find in the literature the assertion of this lemma (in [6] it is stated only for modular lattices). In § 3 of this chapter we shall derive this lemma from a more general theorem, using the deep theorem of ORE (or Lemma 12). We should like to point out that a direct proof of this lemma through Theorem III meets the same difficulty as that mentioned in this paper as Problem 1.

From Lemma 10 it is clear

LEMMA 12. *If an element s is standard in the lattice L as well as in its dual, then s is neutral.*

COROLLARY. *n is neutral if and only if*

$$x\cap(n\cup y)=(x\cap n)\cup(x\cap y),$$
$$x\cup(n\cap y)=(x\cup n)\cap(x\cup y)$$

for all $x, y\in L$.

Thus we see that from the five equalities which we have got from Theorem II to be characteristic for the neutrality of an element, three may be omitted.

LEMMA 13. *Let s and n be elements of L such that n is neutral, $s\leqq n$ and s is standard in $(n]$. Then s is a standard element of L.*

PROOF. From Theorem 6 we know that $x\to(x\cap n, x\cup n)$ is an isomorphism between L and a sublattice of $L_n=(n]\times[n)$. Under this isomorphism $s\to(s, n)$. Since s is standard in $(n]$ and n in $(n]$, therefore s is standard in

L_n. Since the property of being standard is preserved under taking a sublattice and under isomorphism, we get that s is standard in L.

If s is neutral in n, then — it is clear from the above proof — s is neutral in L too, i. e.

COROLLARY 1. *Let s be a neutral element of $(n]$ and n neutral in the lattice L. Then s is a neutral element of L.**

Applying Lemma 13 to $I(L)$ we get

COROLLARY 2. *Let S and N be ideals of the lattice L, $S \subseteq N$ such that N is neutral in L and S is standard in N. Then S is a standard ideal of L.*

Applying Corollary 1 to $I(L)$ we get a theorem of HASHIMOTO [14]:

COROLLARY 3. *A neutral ideal of a neutral ideal is neutral in the whole lattice.*

It is easy to see that the same assertion is not true for standard ideals or elements. As a counterexample take the lattice U and the elements $p > q$. p is standard in U, q is standard in $(p]$, but $(q]$ is not even a homomorphism kernel!

PROBLEM 3. We have seen in the Corollary of Lemma 12 that it is possible to define the neutral elements with the aid of two equalities. Is it possible to define neutrality by a single equality? (E. g.: is the neutrality of n equivalent to the condition that $(x \cap y) \cup (y \cap n) \cup (n \cap x) = (x \cup y) \cap (y \cup n) \cap \cap (n \cup x)$ holds for all $x, y \in L$?)

PROBLEM 4. Let G be a finite group and $L(G)$ the lattice of all subgroups of G. Characterize the standard elements of $L(G)$ (the same problem for neutral elements of $L(G)$ has been solved by G. ZAPPA [35]).

§ 2. Standard elements in weakly modular lattices

Our aim in this section is to prove the coincidence of (distributive and) standard and neutral elements in weakly modular lattices. This theorem contains a part of Lemma 10, that has asserted the same in modular lattices. There the proof was trivial, in consequence of the application of Theorem IV. But in weakly modular lattices we are in lack of a theorem of this kind, therefore the proof is not so simple.

* *Added in proof* (13 February 1961). The following assertion may be proved: Let a and b be neutral elements of the lattice L, $a \leqq b$ and c a standard (neutral) element of b . Then c is a standard (neutral) element of L.

THEOREM 7. *In a weakly modular lattice L, an element d is distributive if and only if it is neutral.*

PROOF. It follows easily from a theorem of ORE's paper [25] that d is distributive if and only if $x \equiv y\ (\Theta[(d]])$ is equivalent to $[(x \cap y) \cup d] \cap \cap (x \cup y) = x \cup y$. It follows that the kernel of the homomorphism induced by the congruence relation $\Theta[(d]]$ is $(d]$. Further, if $x, y \geq d$ and $x \equiv y\ (\Theta[(d]])$, then $x = y$, because $x \cup y = [(x \cap y) \cup d] \cap (x \cup y) = x \cap y$. From these facts we will use only the following:

(*) If $a \leq b \leq d \leq c \leq e$ and d is a distributive element, then $\overline{a, b} \to \overline{c, e}$ implies $c = e$.

Indeed, under the stated conditions, $\overline{a, b} \to \overline{c, e}$ implies $c \equiv e\ (\Theta[(d]])$, and so $c = e$.

Now let d be a distributive element of the weakly modular lattice L. First we prove that d is standard, that is, we prove the validity of (9). Suppose (9) does not hold for a fixed couple $x, y \in L$. Then

$$x \cap (d \cup y) > (x \cap d) \cup (x \cap y).$$

Denote by a the left member of this inequality and by b the right member. We prove that

$$\overline{d, d \cap x} \to \overline{a, b}, \tag{11}$$

namely,

$$\overline{d, d \cap x} \xrightarrow{1} \overline{(d \cup x) \cap (d \cup y), b} \xrightarrow{1} \overline{a, b}.$$

Indeed, because of $d \cap x \leq b$ we have to prove for the validity of

$$\overline{d, d \cap x} \xrightarrow{1} \overline{(d \cup x) \cap (d \cup y), b} \quad \text{only} \quad d \cup b = (d \cup x) \cap (d \cup y).$$

But $d \cup b = d \cup (x \cap d) \cup (x \cap y) = d \cup (x \cap y) = (d \cup x) \cap (d \cup y)$, for d is distributive. Now, using the inequalities $a \leq (d \cup x) \cap (d \cup y)$ and $a > b$, we see that $b = b \cap a$ and $a = (d \cup x) \cap (d \cup y) \cap a$ are trivial. Thus

$$\overline{(d \cup x) \cap (d \cup y), b} \xrightarrow{1} \overline{a, b}$$

and (11) is proved.

Next we verify that

$$\overline{d, d \cup y} \to \overline{a, b}, \tag{12}$$

namely

$$\overline{d, d \cup y} \xrightarrow{1} \overline{d \cap x, a} \xrightarrow{1} \overline{a, b}.$$

To prove the first part of this statement, we have to show only $a \cap d = d \cap x$, but $a \cap d = d \cap x \cap (d \cup y) = d \cap x$. The second part of the assertion is clear.

Let us use the condition $a > b$ and the weak modularity of L: from these it follows the existence of elements u, v for which

$$\overline{a, b} \rightarrow \overline{u, v}, \quad d \leqq v < u \leqq d \cup y. \tag{13}$$

From (11) and (13) it follows $\overline{d, d \cap x} \rightarrow \overline{u, v}$, in contradiction to (*). Thus we have got a contradiction from $a > b$, so $a = b$, i. e. d is standard.

The second step of the proof is: using that d is standard, we prove that it is neutral.

If this statement is not true, then by Lemma 10 we conclude the existence of elements x, y of L such that

$$d \cap (x \cup y) > (d \cap x) \cup (d \cap y),$$

i. e. the condition (i′) of Theorem III does not hold. Putting $s_1 = d \cap (x \cup y)$ and $s_2 = (d \cap x) \cup (d \cap y)$ let us suppose $s_1 > s_2$. First we prove that

$$s_1 \cup x > s_2 \cup x \quad \text{and} \quad s_1 \cup y > s_2 \cup y.$$

Suppose that one of these does not hold, for instance, $s_1 \cup x \mathrel{\not>} s_2 \cup x$; then from $s_1 > s_2$ we have $s_1 \cup x = s_2 \cup x$. We will see that it follows $\overline{d \cap x, x} \rightarrow \overline{s_1, s_2}$, namely

$$\overline{d \cap x, x} \xrightarrow{1} \overline{s_2 \cup (d \cap x), s_2 \cup x} \xrightarrow{1} \overline{s_1, s_2}.$$

To prove this it is enough to show that $s_1 \cap [s_2 \cup (d \cap x)] = s_2$ and $s_1 \cap (s_2 \cup x) = s_1$ Indeed, $s_1 \cap [s_2 \cup (d \cap x)] = s_1 \cap s_2 = s_2$ and $s_1 \cap (s_2 \cup x) = s_1 \cap (s_1 \cup x) = s_1$ (we have used $s_1 \cup x = s_2 \cup x$ in this step). Again from $s_1 > s_2$ and from the weak modularity it follows the existence of elements u, v with $d \cap x \leqq u < v \leqq x$ and $\overline{s_1, s_2} \rightarrow \overline{u, v}$. But $s_1, s_2 \leqq d$, and so $s_1 \equiv s_2$ (Θ_d), consequently $u \equiv v$ (Θ_d). Therefore (see condition (γ) of Theorem 1) $v = u \cup d_1$ with a suitable $d_1 \leqq d$. Then $v = u \cup d_1 \leqq u \cup (d \cap x) = u$, for we get from $v = u \cup d_1$ that $d_1 \leqq v \leqq x$, and hence $d_1 \leqq d \cap x$. The inequality we have just proved is in contradiction to the hypothesis $v > u$. Thus we have proved that $s_1 \cup x > s_2 \cup x$, and in a similar way one can prove $s_1 \cup y > s_2 \cup y$.

Now, using $s_1 \cup x > s_2 \cup x$ and $s_1 \cup y > s_2 \cup y$, we prove that

$$\overline{d \cap (s_2 \cup x), s_2 \cup x} \rightarrow \overline{s_1 \cap (s_2 \cup y), s_1},$$

namely,

$$\overline{d \cap (s_2 \cup x), s_2 \cup x} \xrightarrow{1} \overline{d \cap x, x} \xrightarrow{1} \overline{s_2 \cup y, s_2 \cup (x \cup y)} \xrightarrow{1} \overline{(s_2 \cup y) \cap s_1, s_1}.$$

From these $\overline{d \cap (s_2 \cup x), s_2 \cup x} \xrightarrow{1} \overline{d \cap x, x}$ is clear. To verify $\overline{d \cap x, x} \xrightarrow{1}$ $\xrightarrow{1} \overline{s_2 \cup y, s_2 \cup (x \cup y)}$ we use the inequality $d \cap x \leqq (d \cap x) \cup (d \cap y) = s_2 \leqq s_2 \cup y$, and so $(d \cap x) \cup (s_2 \cup y) = s_2 \cup y$, further $x \cup (s_2 \cup y) = s_2 \cup (x \cup y)$. To prove $\overline{s_2 \cup y, s_2 \cup (x \cup y)} \xrightarrow{1} \overline{(s_2 \cup y) \cap s_1, s_1}$ we have only to observe the inequality $s_1 = d \cap (x \cup y) \leqq s_2 \cup (x \cup y) = x \cup y$, and then $[s_2 \cup (x \cup y)] \cap s_1 = s_1$.

Before applying weak modularity we have to show that $s_1 \neq s_1 \cap (s_2 \cup y)$. Indeed, in case $s_1 = s_1 \cap (s_2 \cup y)$ it follows $s_1 \leqq s_2 \cup y$, and then $s_1 \cup y = s_2 \cup y$, which is a contradiction to $s_1 \cup y > s_2 \cup y$. From this we see that $d \cap (s_2 \cup x) = s_2 \cup x$ is also impossible, for $\overline{d \cap (s_2 \cup x), s_2 \cup x} \to \overline{s_1 \cap (s_2 \cup y), s_1}$, and so $d \cap (s_2 \cup x) = s_2 \cup x$ implies $s_1 \cap (s_2 \cup y) = s_1$. Now, using the weak modularity and $\overline{d \cap (s_2 \cup x), s_2 \cup x} \to \overline{s_1 \cap (s_2 \cup y), s_1}$, it follows the existence of u, v such that $d \cap (s_2 \cup x) \leqq u < v \leqq s_2 \cup x$ and $\overline{s_1 \cap (s_2 \cup y), s_1} \to \overline{u, v}$. It follows now $u \equiv v\ (\Theta_d)$ in a similar way as in the first step of the proof, thus $v = u \cup d'$ $(d' \leq d)$. But from $v \leqq s_2 \cup x$ we have $d' \leqq d \cap (s_2 \cup x)$ for $d \geq s_1 > s_2$. Consequently, $v = u \cup d' \leqq u \cup [d \cap (s_2 \cup x)] = u$, a contradiction to $v > u$.

Thus we have verified the validity of the conditions of Theorem III, thus d is neutral. The proof of Theorem 7 is completed.

COROLLARY 1. *In a weakly modular lattice every standard element is neutral.*

The assertion is clear from condition (δ) of Theorem 1.

Apply this theorem to $I(L)$:

COROLLARY 2. *If $I(L)$ is weakly modular, then any standard ideal of L is neutral.*

COROLLARY 3. *In a relatively complemented lattice L any standard element is neutral.*

COROLLARY 4. *In a modular lattice any standard element and ideal is neutral.*

Corollaries 3 and 4 are immediate consequences of Lemma IV.

Unfortunately, we cannot establish Theorem 7 for distributive ideals, not even the more important Corollary 1 for standard ideals. A detailed discussion of the proof shows that the idea of the proof essentially uses that distributive, resp. standard elements are dealt with and not distributive, resp. standard ideals. It will be clear from § 4 of this chapter that we cannot get the results for ideals by a simple application of Theorem 7 to $I(L)$.

We shall now deal separately with (standard, i. e.) neutral elements of a special class of weakly modular lattices. We intend to show that in relatively complemented lattices the set of all neutral elements is again a relatively complemented lattice. First we prove

LEMMA 14. *Let a, b, c be neutral elements of a lattice L, and suppose $a < b < c$. If a relative complement d of b in the interval $[a, c]$ exists, then it is also neutral and uniquely determined.*

PROOF. We know from Theorem 6 that we can embed L in $L_b = (b] \times [b)$ under the correspondence $x \to (x \cap b, x \cup b)$. Under this $d \to (a, c)$, therefore d

is neutral (for both components of d are neutral) in L_b, and consequently it is neutral in L. The unicity assertion is trivial.

COROLLARY 1. (BIRKHOFF.) *Any complement of a neutral element is neutral.*

COROLLARY 2. *The neutral elements (if any) of a relatively complemented lattice form a relatively complemented distributive sublattice.*

We note that from Corollary 1 we do not get Lemma 14, only that d is neutral in $[a, c]$.

Lemma 14 is not true for standard elements. As an example take the lattice U where o, p, i are standard, while (the unique) relative complement of p in $[o, i]$ is r which is not standard.

PROBLEM 5. Is a distributive (or at least a standard) ideal of a weakly modular lattice neutral?

§ 3. A neutrality condition for standard elements

The last statement of Lemma 10 gives a necessary and sufficient condition for a standard element to be neutral. The condition is not trivial, for it is a conclusion of the comparison of the deep Theorem III with condition (δ) of Theorem 1. But in the previous paragraph, when we wanted to prove the coincidence of standard and neutral elements in weakly modular lattices, we have seen that this condition is not easy to apply. Therefore we set ourselves the aim of finding a sharper condition from which Corollary 1 of Theorem 7 may be easily derived. This is the content of

THEOREM 8. *A standard element s of the lattice L is neutral if and only if $a \geqq b \geqq s \geqq c \geqq e$ and $\overline{a, b} \to \overline{c, e}$ imply $c = e$.*

To prove the "only if" part of the theorem, suppose s is neutral. Then the dual ideal $[s)$ — as an ideal of the dual lattice $\tilde{L}$ — is standard. So it is impossible that a congruence of the form $c \equiv e$ (Θ_s) would hold in the dual lattice $\tilde{L}$, thus $c = e$.

Now we interrupt our proof to observe that the property (*) (stated in the previous section) is characteristic for distributive elements. Indeed, if d is not distributive, then there exist x, y with $d \cup (x \cap y) < (d \cup x) \cap (d \cup y)$. We prove that $d \cup (x \cap y) \equiv (d \cup x) \cap (d \cup y)$ $(\Theta[(d]])$. Indeed, $d \equiv d \cap x \cap y$ $(\Theta[(d]])$. Joining both sides of this relation first with x, then with y, we get

$$d \cup x \equiv x \quad (\Theta[(d]]) \quad \text{and} \quad d \cup y \equiv y \quad (\Theta[(d]]).$$

Meeting the corresponding sides, it results

$$(d \cup x) \cap (d \cup y) \equiv x \cap y \quad (\Theta[(d]])$$

Finally, joining both sides with d, we reach $(d \cup x) \cap (d \cup y) \equiv d \cup (x \cap y)$ $(\Theta[(d]])$, as desired.

Now we prove the "if" part of Theorem 8 If s is not neutral, then s is not distributive in the dual lattice $\tilde{L}$. We may apply the result just obtained to get the existence of $a \leqq b \leqq s \leqq c \leqq e$ in $\tilde{L}$ with $\overline{a, b} \to \overline{c, e}$. This is the same as the required relation in L, completing the proof of Theorem 8.

From the proof we see that the fact that standard elements and not ideals are dealt with, is again very essential.

Suppose that in the lattice L the following condition holds which is a weakened form of weak modularity:

(14) whenever $a > b \geqq c > d$ and $\overline{a, b} \to \overline{c, d}$, then $\overline{c, d} \to \overline{a_1, b_1}$ with suitable elements $a \geqq a_1 > b_1 \geqq b$.

Corollary 1. *If the lattice L satisfies* (14), *then every standard element in L is neutral.*

Proof. Suppose L satisfies (14) and $s \in L$ is standard, but not neutral. Then, owing to Theorem 8 we can find elements $a > b \geqq s \geqq c > d$ such that $\overline{a, b} \to \overline{c, d}$. Now, applying (14), we infer the existence of a pair of elements a_1, b_1 such that $a \geqq a_1 > b_1 \geqq b$ and $\overline{c, d} \to \overline{a_1, b_1}$. Consequently, $a_1 \equiv b_1$ (Θ_s), which is impossible, since $a_1 > b_1 \geqq s$ and s is standard.

Since condition (14) is a generalization of weak modularity, it follows that the last corollary implies Corollary 1 of Theorem 8. We will prove by means of a simple example that this new corollary is stronger than the former one, that is, there exists a not weakly modular lattice L which satisfies (14).

Let L be the lattice defined in § 2 of Chapter II. We adjoin three new elements: $x, 0, 1$, subject to the following relations:

$$x \cup a = 1, \quad x \cap a = 0$$

for all $a \in L$. We get a lattice H whose diagram is given in Fig. 5. In this lattice $\overline{0, x} \to \overline{(2, 0), (1, 0)}$ and despite this fact $\overline{(2, 0), (1, 0)} \to \overline{u, v}$ holds for no $u, v \in L$ for which $x \geqq u > v \geqq 0$. This can be seen from the fact that the two different elements of the interval $[0, x]$ are not congruent modulo $\Theta_{(2,0)(1,0)}$. Consequently, H is not weakly modular. But condition (14) holds in H. Indeed, within L it holds, for L is simple (see Lemma IV). The only remaining case of interest is $1 > a \geqq b > c$ $(a, b, c \in L)$, when $\overline{1, a} \to \overline{b, c}$ always holds. But in this case $\overline{b, c} \to \overline{a, d}$ where d is an arbitrary element with $1 > d \succ a$.

In this counterexample (14) holds and so does the dual of (14). It is easy to show that any counterexample of this kind is infinite.

Lemma 15. *Let L be a semi-discrete lattice in which* (14) *and its dual hold. Then L is weakly modular.*

PROOF. Let $a, b, c, d \in L$, $a > b$, $c > d$ and $d \cap a = b$, $d \cup a \geqq c$. Then $\overline{a, b} \to \overline{c, d}$. If $c = d \cup a$, then simply $\overline{c, d} \xrightarrow{1} \overline{a, b}$. If $c < d \cup a$, we choose an element x with $c \leqq x \prec d \cup a$. Then $\overline{x, d \cup a} \xrightarrow{1} \overline{a, b} \xrightarrow{2} \overline{c, d}$, i.e. $\overline{x, d \cup a} \to \overline{c, d}$ and $d \cup a > x \geqq c > d$. Using (14) we get $\overline{c, d} \to \overline{u, v}$ with suitable $d \cup a \geqq u > > v \geqq x$. But $d \cup a \succ x$, therefore $u = d \cup a$, $v = x$, that is, $\overline{c, d} \to \overline{x, d \cup a}$. Trivially, $\overline{x, d \cup a} \to \overline{a, b}$, and so $\overline{c, d} \to \overline{a, b}$.

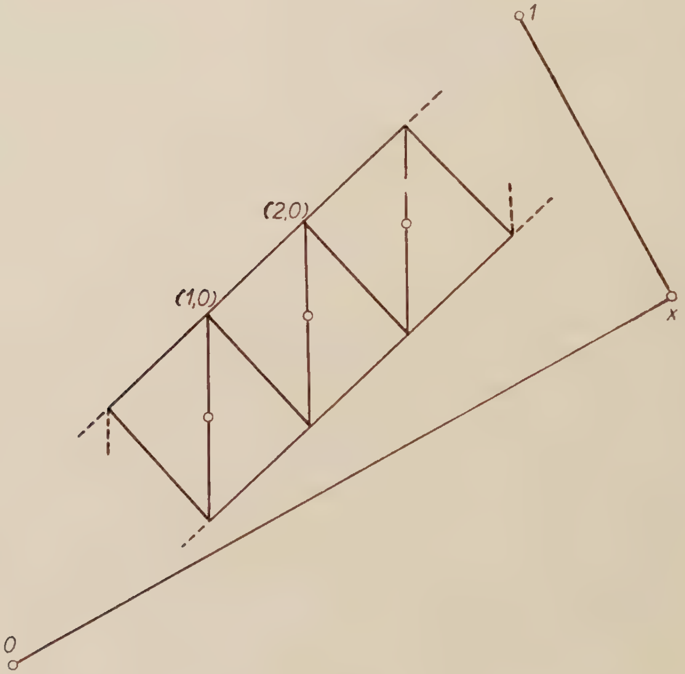

Fig. 5

In case $a > b$, $c > d$, $b \cup c = a$ and $b \cap c \leqq d$, the relation $\overline{a, b} \to \overline{c, d}$ holds, and then we can verify weak modularity by the dual of the above reasoning. The general case $\overline{a, b} \xrightarrow{n} \overline{c, d}$ may be deduced using a simple induction on n.

We see that in Lemma 15, instead of the semi-discreteness of the lattice L, we have used the following weaker property: if $a > b$, then there exist x and y with $a \succ x \geqq b$ and $a \geqq y \succ b$.

PROBLEM 6. Is (14) equivalent to weak modularity in finite lattices?

§ 4. On the lattice of all ideals of a weakly modular lattice

In § 2 the problem arose: why is it not possible to get the results of Theorem 7 for distributive and standard ideals by a simple application of the theorem to $I(L)$. In general, a way of getting a theorem for standard ideals is to prove the same first for standard elements. For instance, we got in this way the coincidence of standard and neutral ideals in modular lattices.

Whenever we make a step of this kind we have to ponder over the question: did we make a supposition on the lattice L which is not preserved if we pass from L to $I(L)$? In case of modular lattices there is no trouble, for if L is modular, then so is $I(L)$. But this is not the case in weakly modular lattices:

THEOREM 9. *The lattice of all ideals of a weakly modular lattice is not necessarily weakly modular.*

PROOF. We have to construct a weakly modular lattice K such that $I(K)$ is not weakly modular. Consider the chain of non-negative integers and take the direct product of this chain by the chain of two elements. The elements of this lattice are of the form $(n, 0)$ and $(n, 1)$, where 0 and 1 are the zero and unit elements of **2** and n is an arbitrary non-negative integer. Further, we define the elements x_n ($n = 1, 2, \ldots$) satisfying the following relations:

$$x_n \cup (n-1, 1) = x_n \cup (n, 0) = (n, 1),$$
$$x_n \cap (n-1, 1) = x_n \cap (n, 0) = (n-1, 0).$$

Thus we have got a lattice L. Finally, we define three further elements $x, y, 1$ subject to

$$\left.\begin{aligned} x \cup y = x \cup z = y \cup z = 1, \\ x \cap y = x \cap z = y \cap z = (0,0) \end{aligned}\right\} \qquad (z \neq 0,\ z \in L).$$

Denote the partially ordered set of all these elements by K. The elements of K are denoted by ∘ in Fig. 6.

It is easy to see that K is a lattice. K is simple and so, by Lemma IV, weakly modular. All but two ideals of K are principal ideals, these exceptional ones are denoted by ⊙ in the diagram, thus the diagram of K, completed by these two elements, gives the diagram of $I(K)$. Now, it is easy to see that K is not weakly modular. Indeed, under the congruence relation generated by the congruence of the two new elements, no two different elements of K are congruent. While from the congruence of any two different elements of K it follows the congruence of the two new elements, we have considered K to be imbedded in $I(K)$. The existence of the lattice K proves Theorem 9.

Some related unsolved problems are listed at the end of this section.

So far we could assure the weak modularity only of the lattice of all ideals of a modular lattice. Naturally, the same is true for every weakly modular lattice in which the ascending chain condition holds, because in this case the lattice of all ideals is identical with (more precisely isomorphic to) the original lattice. The following question arises: is it possible that the lattice of all ideals of a relatively complemented lattice is weakly modular if in the lattice the ascending chain condition does not hold? Is it possible

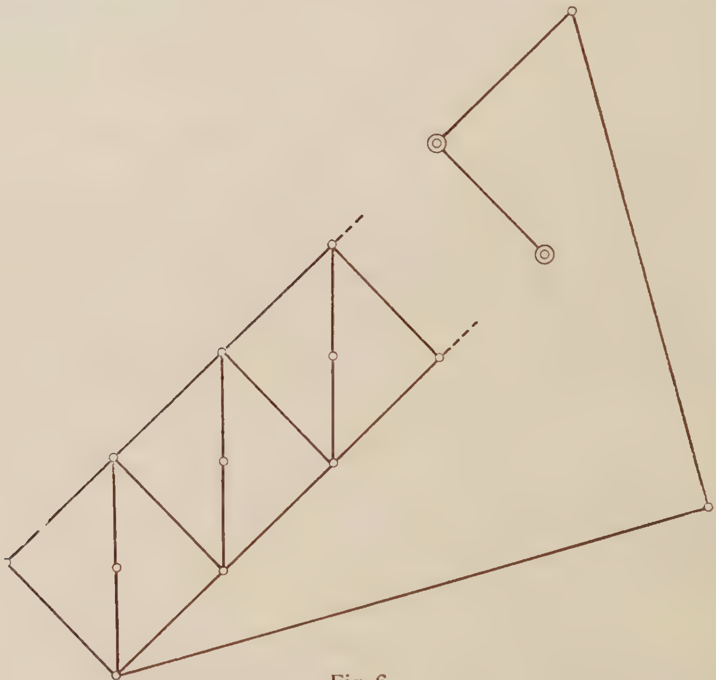

Fig. 6

that the ideal lattice of the same is relatively complemented? The interest of this latter question is that in modular lattices the answer is always negative as a consequence of a theorem of HASHIMOTO [15]. Despite this, the following assertion is true:

There exists a relatively complemented lattice L, not satisfying the ascending chain condition, such that $I(L)$ is relatively complemented. This lattice may be chosen to be semi-modular.

To construct L, consider an infinite set H. We say that the partition p of H, which devides the set H into the disjoint subsets H_α, is finite, if

all but a finite number of the H_α consist of one element, and every H_α consists of a finite number of elements. We denote by $FP(H)$ the set of all finite and by $P(H)$ the set of all partitions of H.

It is clear that the join and meet of any two finite partitions are finite again, and if a partition is smaller than a finite partition, then it is also finite. It follows that $FP(H)$ is an ideal of the lattice $P(H)$. Now, it is easy to prove that just the finite partitions are the elements of the lattice $P(H)$ which are inaccessible from below. Indeed, if p is a finite partition, then the interval $[\omega, p]$ of the lattice $P(H)$ is finite, therefore p is inaccessible from below. Now suppose p is not finite, and let $\{H_\alpha\}$ be the corresponding partition of H (the H_α are pairwise disjoint). Either infinitely many H_α are containing more than one element, or at least one H_α contains an infinity of elements. In the first case, assume that $H_1, H_2, \ldots$ contain more than one element. We define the partition p_i to be the same as p on the set $H \setminus \bigvee_{j=i+1}^{\infty} H_j$, while on the set $\bigvee_{j=i+1}^{\infty} H_j$ let all the classes of p_i consist of one element. Obviously, $p_1 < p_2 < \cdots$ and $\bigvee p_i = p$, consequently, p is accessible from below. In the second case, let H_1 be a set which contains infinitely many elements $\{x_1, x_2, \ldots\}$. We define the partition p_i: upon the set $H \setminus H_1$ it is the same as p, $\{x_1, \ldots, x_i\}$ is one class, and all the x_n $(n > i)$ form separate classes. Again, $p_1 < p_2 < \cdots$ and $\bigvee p_i = p$, so p is accessible from below.

It is also clear that every partition is the complete join of finite partitions and, finally, it is well known (it follows trivially from Lemma III) that $P(H)$ is meet continuous. It follows from a theorem of KOMATU [17] that $P(H)$ is isomorphic to the lattice of all ideals of $FP(H)$.

Now we will prove that $FP(H)$ satisfies the requirements. We have to prove yet that in $FP(H)$ the ascending chain condition does not hold, that $FP(H)$ and $P(H)$ are relatively complemented, and finally that $FP(H)$ is semi-modular. The first of these assertions is trivial, since H is infinite. The second and the third assertions have been proved in [25] for $P(H)$, but these properties are preserved under taking an ideal of the lattice, therefore these hold in $FP(H)$.

We could assure the weak modularity of the ideal lattice of a modular lattice, for the modularity of a lattice may be defined by an equality. We now show that if the weak modularity of a lattice is a consequence of the fulfilment of a system of equalities, then the ideal lattice is also weakly modular. First we prove a general theorem which will serve for other purposes as well.

To formulate the theorem we need two notions. Following ORE [25] we

call a subset $\bar{I}$ of the ideal I a *covering system* of I if $I=\{x; \exists y \in \bar{I}, x \leqq y\}$. Thus, for instance, $\bar{I}=I$ is always a covering system and if $I=(a]$, then $\{a\}$ is a covering system. If I is generated by the set $\{x_\alpha\}$, then the finite joins of the x_α form a covering system.

Let $f_\alpha(y, x_1, \ldots, x_n)$ and $g_\alpha(y, x_1, \ldots, x_n)$ be lattice polynomials, where n depends on α and α runs over an arbitrary set of indices A. (It is not a restriction that $f_\alpha(y, x_1, \ldots, x_n)$ and $g_\alpha(y, x_1, \ldots, x_n)$ depend on the same number of variables. Indeed, if $g_\alpha=g_\alpha(y, x_1, \ldots, x_r)$, $r<n$, then define $g'_\alpha(y, x_1, \ldots, x_n)=g_\alpha(y, x_1, \ldots, x_r) \cup (x_1 \cap x_2 \cap \cdots \cap x_r \cap \cdots \cap x_n \cap y)$. Independently of the values of the $x_1, \ldots, x_n$, the equality $g_\alpha(y, x_1, \ldots, x_r)= =g'_\alpha(y, x_1, \ldots, x_n)$ always holds.) We say that the element s is of the type $f_\alpha=g_\alpha$ $(\alpha \in A)$, if for all $a_1, \ldots, a_n \in L$ and $\alpha \in A$ we have $f_\alpha(s, a_1, \ldots, a_n)= =g_\alpha(s, a_1, \ldots, a_n)$. It is clear from (9) that the standard elements are of the type $f_\alpha=g_\alpha$ with the polynomials $f_1(y, x_1, x_2)=x_1 \cap (y \cup x_2)$ and $g_1(y, x_1, x_2)= =(x_1 \cap y) \cup (x_1 \cap x_2)$ and $A=\{1\}$. Similarly, the neutral elements are also of the type $f_\alpha=g_\alpha$; we get a system of five polynomials from the Corollary of Theorem II and another system consisting of two polynomials from the Corollary of Lemma 12.

THEOREM 10. *Given the ideal I of the lattice L and a covering system $\bar{I}$ of I and the lattice polynomials f_α, g_α $(\alpha \in A)$. If every element of $\bar{I}$ is of the type $f_\alpha=g_\alpha$ $(\alpha \in A)$, then I as an element of $I(L)$ is of the type $f_\alpha=g_\alpha$ $(\alpha \in A)$.*

PROOF. It is enough to prove the theorem for one pair of polynomials $f_\alpha=g_\alpha$. For if the theorem failed to be true, then there would be a pair of polynomials $f=g$ such that I does not satisfy the corresponding equality.

Consider the polynomials f and g, and construct the following subsets of L:

$$F=\{t; t \leqq f(a, j_1, \ldots, j_n), a \in \bar{I}, j_1 \in J_1, \ldots, j_n \in J_n\},$$

$$G=\{t; t \leqq g(a, j_1, \ldots, j_n), a \in I, j_1 \in J_1, \ldots, j_n \in J_n\}$$

where $J_1, \ldots, J_n$ are fixed ideals of L. We prove that F is an ideal. It is enough to prove that $t_1, t_2 \in F$ implies $t_1 \cup t_2 \in F$. Indeed, if $t_1, t_2 \in F$, then there exist $a_i \in I$ and $j_{1,i} \in J_1, \ldots, j_{n,i} \in J_n$ $(i=1,2)$ with

$$t_i \leqq f(a_i, j_{1,i}, \ldots, j_{n,i}).$$

Now choose an element a of $\bar{I}$ for which $a_1 \cup a_2 \leqq a$. Then $f(a, j_{1,1} \cup \cup j_{1,2}, \ldots, j_{n,1} \cup j_{n,2})$ is an element of F, and since the lattice polynomials are isotone functions of their variables, $t_1 \cup t_2 \leqq f(a, j_{1,1} \cup j_{1,2}, \ldots, j_{n,1} \cup j_{n,2})$ is clear, and so $t_1 \cup t_2 \in F$. Similarly, we can prove that G is also an ideal. If $t \in F$, then $t \leqq f(a, j_1, \ldots, j_n)$, but $f(a, j_1, \ldots, j_n)=g(a, j_1, \ldots, j_n)$, for a is

an element of the type $f=g$, and so $t \leqq g(a, j_1, \ldots, j_n)$, that is, $t \in G$. We get $F \subseteq G$ and similarly $G \subseteq F$, that is, $F=G$. Owing to Lemma I, $F=f(I, J_1, \ldots, J_n)$ is clear. $G=g(I, J_1, \ldots, J_n)$ holds as well. Summing up, we got that $f(I, J_1, \ldots, J_n)=g(I, J_1, \ldots, J_n)$ and that was to be proved.

Now we turn our attention to corollaries of this theorem. We say that the lattice L is of the type $f_\alpha=g_\alpha$ if every element of L is of the same type, i.e. if the equalities $f_\alpha=g_\alpha$ $(\alpha \in A)$ identically hold.

COROLLARY 1. *Let* f_α, g_α $(\alpha \in A)$ *be lattice polynomials and suppose* L *is of the type* $f_\alpha=g_\alpha$ $(\alpha \in A)$. *Then this system of equalities holds in* $I(L)$ *too.*

Corollary 1 follows immediately from Theorem 10 taking $\bar{I}=I$ for all ideals $I \in I(L)$.

This corollary was known by BIRKHOFF (see Ex. 1 and Ex. 2 of pages 79 and 80 in [6], especially Ex. 2 (b*)). It answers in affirmative the first question of Ex. 2 (b*).

From Corollary 1 it follows immediately the following assertion, consisting of a theorem of STONE and one of DILWORTH:

COROLLARY 2. *The lattice of all ideals of a modular lattice is modular; the lattice of all ideals of a distributive lattice is distributive.*

Since both the standard and neutral elements are of the type $f_\alpha=g_\alpha$, we get from Theorem 10 the following

COROLLARY 3. *The principal ideal generated by a neutral element is neutral. A standard element generates a standard principal ideal.*

As a generalization of Lemma 13 we get

COROLLARY 4. *Let* N *be a neutral ideal of the lattice* L. *If* F *is an ideal of the type* $f_\alpha=g_\alpha$ $(\alpha \in A)$ *of the lattice* N *and the zero of any lattice is of the type* $f_\alpha=g_\alpha$ $(\alpha \in A)$, *then* F *is an ideal of the same type of the lattice* L.

PROOF. Owing to Theorem 10, we get that $(N]$ is a neutral ideal of $I(L)$ and $(F]$ is an ideal of the type $f_\alpha=g_\alpha$ of the lattice $(N]$. Therefore, it is enough to prove this assertion for a neutral element n and for an element f of type $f_\alpha=g_\alpha$. The proof may be carried out just in the same way as that of Lemma 13, we have to use only the assumption that the zero is of type $f_\alpha=g_\alpha$ (this has been satisfied trivially for standard elements).

We note that the supposition: the zero element of every lattice is of type $f_\alpha=g_\alpha$ $(\alpha \in A)$ is essential. If this does not hold, then the zero element is of type $f_\alpha=g_\alpha$ in the principal ideal generated by the zero element, the zero element is neutral and despite this fact the conclusion of Corollary 4 does not hold.

An interesting special case of Corollary 4 is

COROLLARY 4′. *An element of type $f_\alpha = g_\alpha$ $(\alpha \in A)$ of a neutral ideal N of the lattice L is of type $f_\alpha = g_\alpha$ $(\alpha \in A)$ in the whole lattice.*

Finally, we consider the question: is the converse of Theorem 10 or any of its corollaries true?

The converse of Theorem 10 is not true. It is not true even in the very special case when the defining system of the standard elements is in question. In fact, we have shown in § 2 of Chapter II the existence of standard ideals in lattices without standard elements.

The converses of Corollaries 1 and 2 are naturally true, for L is a sublattice of $I(L)$.

The converse of Corollary 4 states the following:

Assume that an ideal N of the lattice L has the property that whenever F is an ideal of the type $f_\alpha = g_\alpha$ $(\alpha \in A)$ (provided the 0 of any lattice is of this type) in N, then it is of the same type in the whole L. Then N is a neutral ideal.

This assertion is obviously true. Indeed, if an ideal N has the property required, then since N is a neutral ideal of N and since the property of being neutral is a property of type $f_\alpha = g_\alpha$ $(\alpha \in A)$, it follows N is neutral in the whole lattice and this was to be proved.

We now prove that the converse of Corollary 4′ is not true in general. Let L be a lattice generated by the elements x, y, z, $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$. We require that

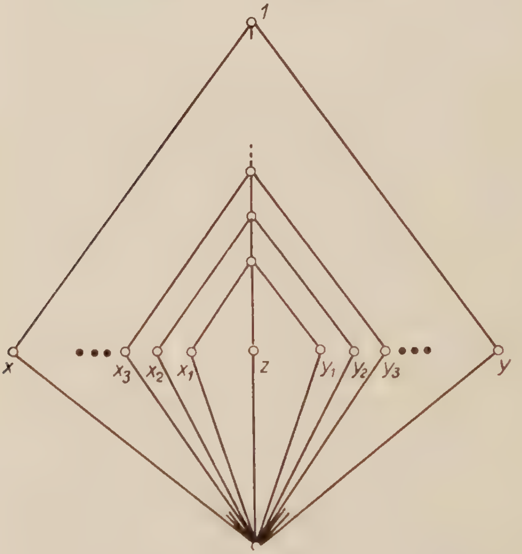

Fig. 7

$$x \cap y = y \cap z = z \cap x = x_i \cap x_j = x_i \cap y_j = y_i \cap y_j = 0 \qquad (i > j)$$

and

$$z < x_1 \cup y_1 < x_2 \cup y_2 < \cdots < x \cup y, \quad z \cup y_i = z \cup x_i = x_i \cup y_i.$$

Let N be the ideal of L generated by the z, x_i and y_i $(i = 1, 2, \ldots)$. The figure of L is shown in Fig. 7. We prove that any element n of the type $f_\alpha = g_\alpha$ of N is of the same type in L. This follows easily from the following

assertion: if L_1 is a sublattice of L generated by n, x and $u_1, \ldots, u_k$, then one can find a sublattice L_2 of N such that $L_1 \simeq L_2$ and under this isomorphism n corresponds to n. The validity of this assertion follows directly from the construction.

It remains to show that N is not neutral. Indeed, $N \cap (x] = (0]$ and $N \cap (y] = (0]$, but $N \cap [(x] \cup (y]] = N$.

PROBLEM 7. Is it possible to construct a relatively complemented lattice L such that $I(L)$ is not weakly modular? (This would be a sharpening of Theorem 9.)

PROBLEM 8. Is any homomorphism kernel of a relatively complemented lattice a neutral ideal?

REMARK. Using Theorem 11 we see that Problem 8 is a special case of Problem 5.

PROBLEM 9. Let the lattice polynomials f_α, g_α ($\alpha \in A$) and f'_β, g'_β ($\beta \in B$) be given. Give condition on the polynomials that an element be of the type $f_\alpha = g_\alpha$ ($\alpha \in A$) if and only if it is of type $f'_\beta = g'_\beta$ ($\beta \in B$).

PROBLEM 10. Give types of weakly modular lattice which are defined by identical relations and are different from the following three classes of lattices: a) the class consisting only of the lattice of one element; b) the class of distributive lattices; c) the class of modular lattices.

REMARK. BIRKHOFF states in [6] that among the modular lattices generated by three elements, one can define with the aid of identical relations only the above listed three classes of lattices.

PROBLEM 11. Find identities (in the variables s, x, y) ensuring that in the lattice generated by s, x and y the element s should be standard.

REMARK. The same problem for neutral elements is solved by Corollary of Theorem II.

PROBLEM 12. Determine the free standard lattice $FSL(3)$, that is, the free lattice generated by the elements s, x, y and we suppose s to be standard in $FSL(3)$.

REMARK. The same problem for neutral elements has been solved in [6], for the free neutral lattice with three generators is the free distributive lattice with three generators.

CHAPTER IV

HOMOMORPHISMS AND STANDARD IDEALS

§ 1. Homomorphism kernels and standard ideals

It is assured already by Lemma 3 that a standard ideal is a homomorphism kernel. The converse statement — as we have remarked — is not true in general, not even in modular lattices.[6] A simple example for that is shown in Fig. 8. The principal ideal $(a]$ of this lattice is a homomorphism kernel (obviously, because it is a prime ideal), but it is not standard for $x \cap (a \cup t) = x$ but $(x \cap a) \cup (x \cap t) = y$.

Now we prove:

THEOREM 11. *Let L be a section complemented lattice. Then every homomorphism kernel of L is a standard ideal and every standard ideal is the kernel of precisely one congruence relation.*

PROOF. Let the ideal I of the lattice L be the kernel of the homomorphism induced by the congruence relation Θ. Let $a \equiv b$ (Θ), $a \geqq b$, $a, b \in L$. We pick out an arbitrary element u of I. From the definition of section complementedness it follows the existence of an element $c \leqq u \cap a \cap b$ of L, for which the dual ideal $[c)$ as a lattice is section complemented, that is, any interval of type $[c, d]$ is complemented. From $u \in I$ it follows $c \in I$. Let b' be the relative complement of b in the interval $[c, a]$. From $a \equiv b$ (Θ) we conclude that $c = b \cap b' \equiv a \cap b' = b'$ (Θ), and, since I is a homomorphism kernel, $b' \in I$. Then $b \cup b' = a$ and $b' \in I$, thus by condition (γ'') of Theorem 2, I is a standard ideal.

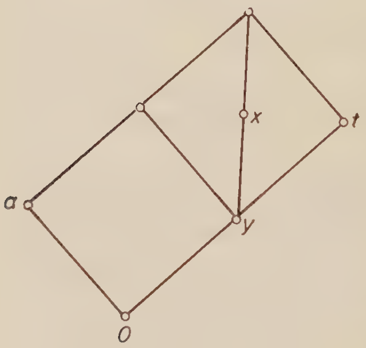

Fig. 8

At the same time we have proved that if I is the kernel of the homomorphism induced by Θ, then $\Theta = \Theta[I]$, and it follows that every standard ideal is the kernel of at most one homomorphism. It is known already from Lemma 3 that every standard ideal is the kernel of at least one homomorphism. Thus the proof of Theorem 11 is completed.

[6] It is included in a theorem of HASHIMOTO [14] that in a finite modular lattice every homomorphism kernel is standard and every congruence relation is a standard one if and only if the lattice is a direct product of simple lattices.

Theorem 11 is not true with "neutral ideal" instead of "standard ideal". A counterexample is shown in Fig. 9. That lattice is section complemented, the principal ideal $(a]$ is standard, but not neutral.

Since relatively complemented lattices form an important subclass of section complemented lattices (see Lemma V), therefore we formulate the assertions of Theorem 11 again for relatively complemented lattices.

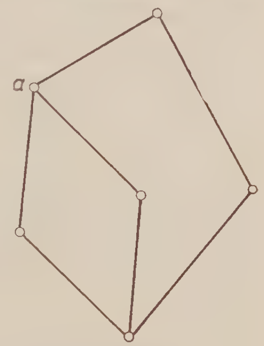

Fig. 9

COROLLARY 1. *In relatively complemented lattices there is a one-to-one correspondence between the standard ideals and the homomorphisms having kernel, letting a homomorphism correspond to its kernel.*

From Lemma 10 we know that in a modular lattice every standard ideal is neutral. Hence we get

COROLLARY 2. *In section complemented modular lattices there is a one-to-one correspondence between the neutral ideals and the homomorphisms having kernel.*

It is obvious that every homomorphism of the lattice L has a kernel if (and only if) L has a zero. Adding this trivial remark to Corollary 2 we get

COROLLARY 3. *In relatively complemented modular lattices with zero there is a one-to-one correspondence between the neutral ideals and homomorphisms, letting a homomorphism correspond to its kernel. This correspondence is an isomorphism between the lattice of all neutral ideals of L and $\Theta(L)$.*

This Corollary 3 is BIRKHOFF's theorem mentioned in the Introduction. We did not use in the formulation the expression "section complemented", for in modular lattices it means the same as "relatively complemented". This corollary shows that Theorem 11 is actually a generalization of BIRKHOFF's theorem.

Using Corollary 2 of Theorem 7, we can give another generalization of BIRKHOFF's theorem:

COROLLARY 4. *Let L be a lattice with zero, and suppose that $I(L)$ is weakly modular. Then there is a natural one-to-one correspondence between the neutral ideals and homomorphisms of L.*

From Theorem 11 and its Corollary 4 we shall get a generalization as well as a new proof of the structure theorem of relatively complemented lattices. An important tool in proving the structure theorem was the assertion that any two congruence relations of a relatively complemented lattice are permutable. First we generalize this assertion:

Corollary 5. *Any two congruence relations of a section complemented lattice L are permutable.*

Proof. Suppose the assertion is not true. Then (as it is obvious from the proof of Lemma 7) there exist $a>b>c$ $(a, b, c \in L)$ and $\Theta, \Phi \in \Theta(L)$ such that $a \equiv b$ (Θ), $b \equiv c$ (Φ), while no element d of $[c, a]$ satisfies $a \equiv d$ (Φ) and $d \equiv c$ (Θ).

By the definition of section complementedness, there exists an $e \in L$ such that $a, b, c \in [e)$ and $[e)$ is section complemented. Any congruence relation Θ of L induces in $[e)$ a congruence relation $\bar{\Theta}$: $x \equiv y$ $(\bar{\Theta})$ $(x, y \in [e))$ if and only if $x \equiv y$ (Θ). $[e)$ is a section complemented lattice with zero, therefore every congruence relation of $[e)$ is a standard congruence relation. Owing to Lemma 7 we see that $\bar{\Theta}$ and $\bar{\Phi}$ are permutable, that is, there exists $d \in [c, a]$ with $a \equiv d$ $(\bar{\Phi})$ and $d \equiv c$ $(\bar{\Theta})$. This implies $a \equiv d$ (Φ) and $d \equiv c$ (Θ), and this contradiction (the existence of such a d) proves the validity of Corollary 5.

Remark. In applications it is enough to have Corollary 5 for section complemented lattices with zero. In this special case all the proof of Corollary 5 is the following: from the supposition and Theorem 11 it follows that every congruence relation of L is a standard congruence relation and by Lemma 7 any two standard congruence relations are permutable. — We remark that in proving Corollary 6 we shall not use Corollary 5.

Dilworth has stated a structure theorem for relatively complemented lattices with ascending chain condition and with zero. We will suppose only that the lattice is section complemented and satisfies the ascending chain condition, and in this case we shall give a necessary and sufficient condition for the validity of the structure theorem.

Corollary 6. *Let L be a section complemented lattice with zero satisfying the ascending chain condition. A necessary and sufficient condition for L to be the direct product of simple lattices is that L be weakly modular.*

Proof. First, suppose L is weakly modular. If L is not simple, there is a non-trivial congruence relation Θ of L, and let $I = \{x; x \equiv 0 \ (\Theta)\}$. From the ascending chain condition it follows that $I(L) \sim L$ and I is principal, $I = (n]$. Thus $I(L)$ is weakly modular, hence from Corollary 4 of Theorem 11 it follows that $(n]$ is neutral. From Corollary 3 of Theorem 10 we get the neutrality of n. From the ascending chain condition it follows the existence of maximal neutral elements n'_α $(\alpha \in A)$ different from 1. The complements n_α $(\alpha \in A)$ of the n'_α are minimal neutral elements of L. The index set A is finite. Otherwise, let $n_1, n_2, \ldots$ be distinct minimal neutral elements. Then

putting $m_j = \bigcup_{i=1}^{j} n_i$, we have $0 < m_1 < m_2 < \cdots$ contradicting the ascending chain condition. Let $n_1, \ldots, n_k$ be the minimal neutral elements of L, and put $n = \bigvee_{i=1}^{k} n_i$. If $n \neq 1$, there is a neutral element n'_α with $n \leqq n'_\alpha < 1$. But $(n'_\alpha)' = n_i$ for some i, thus $(n'_\alpha)' \leqq n$, a contradiction. Thus $\bigvee n_i = 1$, that is (Corollary of Theorem 6), $L = (n_1] \times \cdots \times (n_k]$. We have to prove yet that the $(n_i]$ are simple. Indeed, if this does not hold, say $(n_1]$ is not simple, then by the same argument as above, we get the existence of a neutral element n of $(n_1]$ with $0 < n < n_1$. But then (Corollary 3 of Lemma 13) n is neutral in L, contradicting the minimality of n_1.

Conversely, suppose that L is the direct product of simple lattices. By Lemma IV, simple lattices are weakly modular, and it is obvious from the definition of weak modularity that the direct product of a finite number of weakly modular lattices is weakly modular again. Thus L is weakly modular, completing the proof.

As a further application of Theorem 11 we now prove a generalization of a theorem of Shih-Chiah Wang mentioned in the Introduction.

Theorem 12. *Let L be a relatively complemented lattice with* 0 *and* 1. *$\Theta(L)$ is a Boolean algebra if and only if every standard ideal of L is principal.*

Proof. Suppose every standard ideal of L is principal. It follows that every congruence relation of L is of the form Θ_s where s is a standard element. Indeed, every congruence relation Θ is of the form $\Theta = \Theta[S]$, where S is the kernel of the homomorphism induced by Θ, every homomorphism kernel is standard (these assertions are consequences of Theorem 11), and finally, every standard ideal is generated by one element $S = (s]$. From Corollary 1 of Theorem 7 it follows that s is a neutral element. L is section complemented, therefore s has a complement t. By Corollary 1 of Lemma 14, t is also neutral. Using Theorem 3 we get $\Theta_s \cap \Theta_t = \Theta_{s \cap t} = \Theta_0 = \omega$ and $\Theta_s \cup \Theta_t = \Theta_{s \cup t} = \Theta_1 = \iota$, that is, Θ_t is the complement of Θ_s. Therefore, every congruence relation of $\Theta(L)$ has a complement, that is, $\Theta(L)$ is a Boolean algebra.

Conversely, suppose that $\Theta(L)$ is a Boolean algebra. Owing to Theorem 11 we see that every congruence relation of L is of the form $\Theta[S]$ where S is a standard ideal. Let the congruence relation $\Theta[T]$ be the complement of $\Theta[S]$ (T is also a standard ideal). From Theorem 3 it follows $\Theta[S \cap T] = \omega$, $\Theta[S \cup T] = \iota$. Again from Theorem 3 we get $S \cap T = (0]$ and $S \cup T = L$. L has a unit element, therefore both $S \cap T$ and $S \cup T$ are principal ideals and, consequently, from the Corollary of Lemma 8 we obtain the

result: S and T are principal ideals. We have proved that every standard ideal of L is principal, and thus the proof of Theorem 12 is completed.

We know (Lemma 10) that in modular lattices every standard ideal is neutral, consequently, from Theorem 12 we get as a special case the theorem of S. WANG:

COROLLARY. *The lattice of all congruence relations of a complemented modular lattice is a Boolean algebra if and only if every neutral ideal is principal.*

PROBLEM 13. Describe those finite lattices in which we get a one-to-one correspondence between the homomorphisms and standard ideals, letting a homomorphism correspond to its kernel.

REMARK. The lattice of Fig. 7 shows that such a lattice is not necessarily the direct product of simple lattices.

§ 2. Isomorphism theorems

In this section we will show that both isomorphism theorems are true for standard ideals.

THEOREM 13. (First isomorphism theorem for standard ideals.) *Let L be a lattice, S a standard ideal and I an arbitrary ideal of L. Then $S \cap I$ is a standard ideal of I and*

$$(I \cup S)/S \sim I/(I \cap S).$$

PROOF. Corollary of Lemma 9 is just the first assertion of our theorem. The simplest mean to prove the isomorphism statement is the use of the first general isomorphism theorem of RÉDEI [29] (see Ch. I, § 2). We have only to prove that every congruence class of the lattice $I \cup S$ may be represented by an element of I. Indeed, any element x of $I \cup S$ is of the form $y \cup s$ where $s \in S$ and $y \in I$ (see condition (β') of Theorem 2). Further, $x = y \cup s \equiv y(\Theta\ [S])$, and so the congruence class that contains x may be represented by $y \in I$.

According to Theorem 4, the isomorphism statement of the first isomorphism theorem is equivalent to the isomorphism of the intervals $[S, I \cup S]$ and $[I \cap S, I]$ of $I(L)$. We can add to the isomorphism statement the following

SUPPLEMENT. *Let L be a lattice and S a standard ideal of L. Then*

$$[I \cap S, I] \sim [S, I \cup S]$$

for all $I \in I(L)$. An isomorphism is given by the correspondence

$$X \to X \cup S \qquad (X \in [I \cap S, I]).$$

The inverse correspondence is

$$Y \to Y \cap I \qquad (Y \in [S, I \cup S]).$$

PROOF. From condition (δ') (i) of Theorem 2 we get that $X \to X \cup S$ ($X \in [I \cap S, I]$) is a homomorphism. If $X_1, X_2 \in [I \cap S, I]$, then $S \cap X_1 = S \cap X_2$ (for $X_1, X_2 \subseteq I$, and so $S \cap X_i = S \cap X_i \cap I = S \cap I$, $i = 1, 2$). Thus from condition (δ') (ii) of Theorem 2 we get that $S \cup X_1 \neq S \cup X_2$. Therefore, $X \to X \cup S$ is an isomorphism of $[I \cap S, I]$ *into* $[S, S \cup I]$. We prove that

$$(Y \cap I) \cup S = Y \qquad (Y \in [S, I \cup S]),$$

and this will prove that $Y \to Y \cap I$ ($Y \in [S, I \cup S]$) is the inverse of $X \to X \cup S$ ($X \in [I \cap S, I]$), and the latter correspondence maps $[I \cap S, I]$ *onto* $[S, I \cup S]$. Indeed, using condition (δ') (i) of Theorem 2, we get

$$(Y \cap I) \cup S = (Y \cup S) \cap (I \cup S) = Y \cap (I \cup S) = Y,$$

and this completes the proof of the Supplement.

In the last proof we have got a new proof of the isomorphism theorem. In case S is a principal ideal, the isomorphism given in the Supplement is essentially the isomorphism of the two factor lattices.

We remark that HASHIMOTO has got the first isomorphism theorem under the condition that both S and I are neutral ideals.

THEOREM 14. (Second isomorphism theorem for standard ideals.) *Let L be a lattice, S an ideal and T a standard ideal of L, $S \supseteq T$. Then S is standard if and only if S/T is standard in L/T, and in this case*

$$L/S \cong (L/T)/(S/T).$$

PROOF. If S is standard, then from Lemma 6 we get that S/T is standard in L/T. Conversely, suppose S/T is standard in L/T. We show condition (γ'') of Theorem 2 holds for S. We have seen in the proof of Theorem 1, in the step "(β) implies (γ)", that it is enough to prove that $x \equiv y$ ($\Theta[S]$) and $x \geqq y$ imply $x \cap u \equiv y \cap u$ ($\Theta[S]$) for all $u \in L$. (Here $\Theta[S]$ denotes the relation defined in condition (γ'') of Theorem 2.) We denote by $\bar{a}$ the image of the element a under the homomorphism $L \sim L/T$. Then we have $\bar{x} \equiv \bar{y}(\Theta[S/T])$, and since S/T is standard in L/T, therefore with a suitable $\bar{s} \in S/T$ we get $\bar{x} \cap \bar{u} = (\bar{y} \cap \bar{u}) \cup \bar{s}$. Further, since T is standard in L, we can find a $t \in T$ such that $x \cap u = [(y \cap u) \cup s] \cup t$. We put $s_1 = s \cup t$ and get $x \cap u = (y \cap u) \cup s_1$, $s_1 \in S$. This proves that S is standard.

We remark that during the proof we have made effective use of the fact that the congruence classes of L/T under $\Theta[S/T]$ are the homomorphic images of those of L under $\Theta[S]$.

The isomorphism is again proved simply by reference to the general second isomorphism theorem of RÉDEI [29].

COROLLARY. *There is a natural isomorphism*

$$X \rightarrow X/T$$

between the dual ideal $[T)$ *of* $I(L)$ *and* $I(L/T)$, *and this isomorphism makes the standard ideals of* L *containing* T *correspond to the standard ideals of* L/T.

The second isomorphism theorem of HASHIMOTO supposes that S and T are neutral and it is confined to the isomorphism statement.

The question naturally arises: what happens with the neutral ideals under the correspondence of the Corollary of Theorem 14? The answer is: neutral ideals of $[T)$ are mapped upon neutral ideals, but the converse is not true. Of course, T/T is always neutral in L/T, thus, if T is not neutral, then we have found a not neutral ideal carried into a neutral one.

And what happens if T is neutral? An answer is given by

THEOREM 15. (Second isomorphism theorem for neutral ideals.) *Let* L *be a lattice,* S *an ideal,* T *a neutral ideal of* L, $S \supseteq T$. *The ideal* S *is neutral if and only if* S/T *is neutral in* L/T. *In this case we have the isomorphism*

$$L/S \simeq (L/T)/(S/T).$$

SUPPLEMENT. *The ideal* S $(S \supseteq T)$ *is of type* $f_\alpha = g_\alpha$ $(\alpha \in A)$ *(it is presupposed that the zero of any lattice is of this type) if and only if* S/T *is of type* $f_\alpha = g_\alpha$ $(\alpha \in A)$ *in* L/T.

PROOF. Owing to Corollary 3 of Theorem 10 we can reduce the theorem to the case of neutral elements. Then, using Theorem 6, we trivially get (supposing n is neutral) that an element $s \geqq n$ of this lattice is neutral if and only if s is neutral in the lattice $[n)$. On the other hand, $[n)$ is isomorphic to the factor lattice $L/(n]$ and thus the assertion is proved. The assertions for the elements of type $f_\alpha = g_\alpha$ $(\alpha \in A)$ are to be proved in the same way.

We see that the second isomorphism theorem for standard ideals is characteristic for standard ideals. A slightly weaker form of Theorem 14 is already not characteristic for them. Namely:

There exists a lattice L *and a homomorphism kernel* S *of* L *such that* S *is not standard, but if* T *is a homomorphism kernel of* L *with* $T \subseteq S$, *then* S/T *is a standard ideal of* L/T *and*

$$L/S \simeq (L/T)/(S/T).$$

To construct such a lattice we take the chain of non-positive integers, and form the direct product of this chain with the chain of two elements.

The elements of this direct product are of the form $(n, 0)$ and $(n, 1)$ (n is a non-positive integer). We define one further element x subject to

$$\left.\begin{aligned} &x \cup (0,0) = x \cup (-n,1) = (0,1),\\ &x \cap (0,0) = (-1,0),\\ &x \cap (-n,1) = (-n,0) \end{aligned}\right\} \qquad (n = 1, 2, \ldots).$$

The diagram of this lattice is shown by Fig. 10. Let $S = (s]$. By an easy discussion one can show that the requirements are fulfilled.

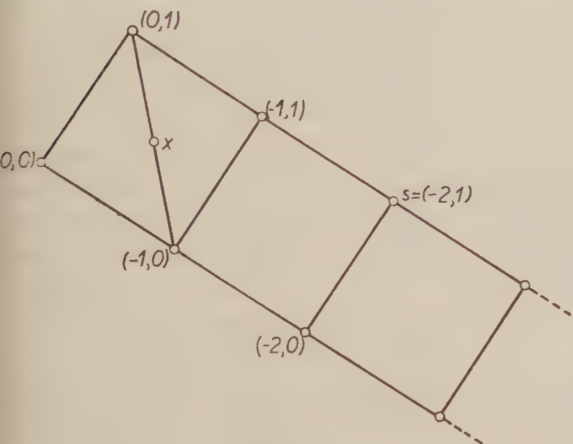

Fig. 10

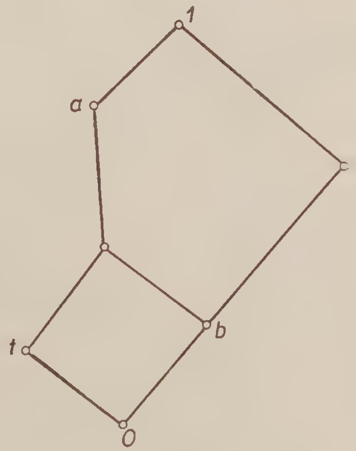

Fig. 11

A very difficult question is: which class of ideals is characterized by the first isomorphism theorem? It is easy to prove that this class is wider than the class of standard ideals. Before giving the example, we first give a precise meaning of the expression: an ideal satisfies the first isomorphism theorem.

We say that the ideal T of the lattice L satisfies the first isomorphism theorem if

$$T \cup I / T \cong I / I \cap T$$

for all $I \in I(L)$.

Now consider the following lattice of Fig. 11. The ideal $(t]$ of this lattice satisfies the first isomorphism theorem, but it is not standard. In this lattice the following ideals satisfy the first isomorphism theorem: $(0]$, $(t]$, $(b]$, $(a]$, $(1]$. These do not form a sublattice of $I(L)$. It is a bit more difficult, but still possible to give such a counterexample, where the ideals satisfying the first isomorphism theorem do form a sublattice of the lattice of ideals.

In some special class of lattices we shall examine the ideals satisfying the first isomorphism theorem in Chapter VI.

Now we are in the position to show how the dictionary, mentioned in the Introduction, works. Let us repeat first the dictionary:

$$\begin{aligned} \text{subgroup} &\to \text{ideal} \\ \text{invariant subgroup} &\to \text{standard ideal} \\ \text{factorgroup} &\to \text{factor lattice modulo a standard ideal} \\ \text{group operation} &\to \text{join.} \end{aligned}$$

Consider the isomorphism theorems: we see that these are word by word translations of the corresponding statements of group theory. To get further examples, consider the following assertion of group theory:

The subgroup N of the group G is invariant if and only if NH is a subgroup for every subgroup H of G.

Here NH is the complex product of N with H. The corresponding notion is the "complex join" of two ideals I, K, let us denote it by $I \vee K$ (the notation $I \cup K$ would be ambiguous, it is used to denote the ideal-theoretical join of I and K), that is, $I \vee K = \{x; x = i \cup k; i \in I, k \in K\}$. Now the "translation" of the above theorem is:

The ideal S of the lattice L is standard if and only if $S \vee K$ is an ideal of L for every ideal K of L.

This theorem is actually true, for it is nothing else but a reformulation of condition (β') of Theorem 2.

The fact that one can use the dictionary so fruitfully seems to prove not only the applicability of the notion of standard ideals, but at the same time the usefulness of the definition of factor lattice.

The existence of the dictionary suggests the idea of possibility to define for general algebras a notion of "standard subalgebra", which is a common generalization of ideals in rings, of invariant subgroups in groups and standard ideals in lattices, and for which one can prove those theorems which can be translated from group theory to lattice theory. In fact, this can be done if we confine ourselves only to the isomorphism theorems and do not consider the result concerning complex product of subgroups and the theorem of § 4 of this chapter. So far we did not succeed in finding any notion which fulfils all the requirements.

§ 3. The Zassenhaus lemma

In groups as well as in rings one can prove the Zassenhaus lemma using the two isomorphism theorems. So it is not surprising that translating the Zassenhaus lemma to lattices, we can prove it without any difficulty.

THEOREM 16. (The Zassenhaus lemma.) *Let I and K be ideals of the lattice L. Further, let S and T be standard ideals of the lattices I and K, respectively. Then $S \cup (I \cap T)$ is a standard ideal of the lattice $S \cup (I \cap K)$ and $T \cup (S \cap K)$ is that of $T \cup (I \cap K)$. Finally, we have the isomorphism:*

$$[S \cup (I \cap K)]/[S \cup (I \cap T)] \simeq [T \cup (I \cap K)]/[T \cup (S \cap K)].$$

PROOF. (The situation is shown by Fig. 12.) Owing to the first isomorphism theorem we have

$$(*) \qquad [S, S \cup (I \cap K)] \simeq [S \cap K, I \cap K].$$

From Lemma 9 it follows that $S \cap K$ and $I \cap T$ are standard ideals of the lattice $I \cap K$, and so from Theorem 3 we get that their join $(S \cap K) \cup (I \cap T)$ is likewise a standard ideal of the same lattice. Consequently, by Lemma 6, $(S \cap K) \cup (I \cap T)/S \cap K$ is a standard ideal of $I \cap K/S \cap K$. From the Supplement of the first isomorphism theorem we know that the isomorphism $(*)$ may be set up by the correspondence $X \to X \cup S$ $(X \in [S \cap K, I \cap K])$. This carries the ideal $(S \cap K) \cup (I \cap T)$ into the ideal $S \cup (I \cap T)$ and $S \cap K$ into S. Thus we get from the above statement that $S \cup (I \cap T)/S$ is a standard ideal of $S \cup (I \cap K)/S$. From the first assertion of the second isomorphism theorem we get that $S \cup (I \cap T)$ is standard in $S \cup (I \cap K)$ and thus the first assertion of the Zassenhaus lemma is proved.

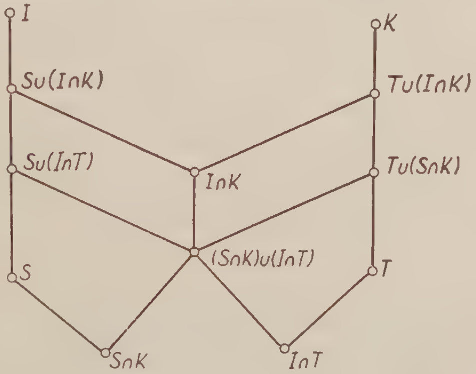

Fig. 12

It follows from the above statements that

$$[S \cup (I \cap K)/S]/[S \cup (I \cap T)/S] \simeq [I \cap K/S \cap K]/[(S \cap K) \cup (I \cap T)/S \cap K],$$

and from the second isomorphism theorem we get

$$[S \cup (I \cap K)]/[S \cup (I \cap T)] \simeq [I \cap K]/[(S \cap K) \cup (I \cap T)].$$

Similarly,

$$[T \cup (I \cap K)]/[T \cup (S \cap K)] \simeq [I \cap K]/[(S \cap K) \cup (I \cap T)],$$

and these two isomorphisms together prove the required isomorphism statement.

From Theorem 4 it follows that the isomorphism statement of the Zassenhaus lemma is equivalent to the isomorphism of the following two intervals of $I(L)$:

$$[S \cup (I \cap T), S \cup (I \cap K)] \simeq [T \cup (S \cap K), T \cup (I \cap K)].$$

SUPPLEMENT. *An isomorphism of these two intervals is given by the correspondence*

$$X \to (X \cap K) \cup T \qquad (X \in [S \cup (I \cap T), S \cup (I \cap K)])$$

whose inverse is

$$Y \to (Y \cap I) \cup S \qquad (Y \in [T \cup (S \cap K), T \cup (I \cap K)]).$$

A special case of some interest of the first assertion of the Zassenhaus lemma is the following:

COROLLARY. *Let A and B be elements of the lattice L and suppose $A > B$. If a is a standard element of the lattice $(A]$ and b is a standard element of the lattice $(B]$, then $a \cup b$ is a standard element of $(a \cup B]$.*

This corollary is a generalization of a part of Theorem 3, which asserts that the join of two standard elements is again a standard element. It would be interesting to give a direct proof of the Corollary.

The proof of the supplement may be got simply by applying twice the supplement of the first isomorphism theorem.

If both S and T are principal ideals, then the isomorphism of the supplement is the isomorphism of the two factor lattice.

The Jordan—Hölder—Schreier theorem follows from the Zassenhaus lemma as usually. To formulate the theorem we need the usual notions:

A chain

$$I = I_0 \supseteq I_1 \supset \cdots \supset I_n = J$$

of the ideals of the lattice L is called a standard series of length n of the interval $[J, I]$ if I_k is standard in I_{k-1} for $k = 1, 2, \ldots, n$. A standard series is called proper if we have always $\supset$ instead of $\supseteq$. A composition series is a proper standard series which has no proper refinement in the interval $[J, I]$.

THE JORDAN—HÖLDER—SCHREIER—ZASSENHAUS THEOREM. *Any two standard series of the interval $[J, I]$ have refinements such that the lengths of the two refined series are the same and the factor lattices of the two refined series are — disregarding of the order — pairwise isomorphic. Further, if in an interval there exists a composition series, then every standard series may be refined to a composition series. Consequently, any two composition series have the same length.*

Obviously, we may make the refinement in the same fashion as in groups and rings. Therefore we may omit the details.

We note that if we defined the standard series, requiring that every member of the series be a standard ideal in the whole lattice, then our Jordan—Hölder—Schreier—Zassenhaus theorem would be a consequence of the general Jordan—Hölder—Schreier—Zassenhaus theorem (see BIRKHOFF [6]). Indeed, in this case from Lemma 7 it follows that any two congruence rela-

tions in question are pairwise permutable. Furthermore, it is not a restriction to suppose that the lattice L has a zero, for we may pass from L to L/J where J is the least standard ideal which occurs.

In [14] HASHIMOTO has proved the above theorem, supposing all the ideals which occur are neutral. From the above remarks it follows that the results of HASHIMOTO are special cases of the theorem of [6].

§ 4. The Schreier extension problem for lattices

We will now examine a lattice-theoretical analogue of the well-known Schreier problem of the theory of groups.

The original problem is the following:

Let the groups G_1 and G_2 be given. Describe all groups G containing G_1 as an invariant subgroup such that $G/G_1 \simeq G_2$.

Translating this problem with the aid of the "dictionary" to lattice theory, we get the following problem:

Let the lattices L_1 and L_2 be given. Describe all lattices L containing L_1 as a standard ideal such that $L/L_1 \simeq L_2$.

Such a lattice will be called the Schreier extension of L_1 by L_2. Schreier extensions always exist, e. g. the direct product of the two given lattices.

SCHREIER's extension problem in groups and rings may be solved by means of certain functions. It is possible to define these functions as a consequence of the invertibility of the group operation. Although the problem is solved for semigroups too — with a suitable definition of invariant subsemigroup — but in this case the requirement is essentially the regularity of the operation with respect to the invariant subsemigroup. Because there is no possibility of defining regularity of the lattice operations, therefore we have to give up the hope of finding a solution similar to the method of the Schreier's functions. This is suggested already by a result of SZÁSZ [33], according to which the rather general notion "das schiefe Product" of RÉDEI ([26] and [27]), including in case of groups all Schreier's extensions, gives in lattices the direct product and nothing more.

Therefore, we had to recourse to other methods.

THEOREM 17. *Given the lattices L_1 and L_2, suppose L_1 has a unit and L_2 has a zero element. L is a Schreier extension of L_1 by L_2 if and only if L is isomorphic to a meet-sublattice*[7] *of $L_1 \times L_2$ which contains all the elements of the form $(x, 0)$ $(x \in L_1)$ and $(1, y)$ $(y \in L_2)$.*

[7] A subset H of the lattice L is called a meet-sublattice if it is closed under forming finite meet and under the partial order induced by that of L it is a lattice. This means that the join in H is not necessarily the same as in L.

Thus, while the group-theoretical Schreier extension problem is reduced to finding certain functions, now the same problem for lattices is reduced to finding certain meet-sublattices of a well-constructed lattice.

Proof. Suppose L is a Schreier extension of L_1 by L_2. Let s be the greatest element of the ideal L_1 of L and embed L in L_s (see the definition of L_s before Theorem 6) in the natural way, that is, with the correspondence $x \to (x \cap s, x \cup s)$. It follows from Theorem 6 that it is a meet isomorphism. The image of L in L_s contains all the elements of the form $(x, 0)$ and $(1, y)$. Because of $L_1 \sim \{a; a \in L, a = (x, 0)\}$ and $L_2 \sim \{a; a \in L, a = (1, y)\}$ we get that L fulfils the requirements of Theorem 17.

Now we suppose that L is a meet-sublattice of $L_1 \times L_2$, and it contains the elements of the form $(x, 0)$ and $(1, y)$. We shall prove that $S = \{a; a \in L, a = (x, 0)\}$ is a standard ideal of L. Consider that partition of L which is induced by the congruence relation $\Theta[S]$ of $L_1 \times L_2$. If we prove that this partition is compatible, then it follows that S is a standard ideal. It is enough to show that if $y = x \vee s, s \in S$ ($\vee$ denotes the join in L), then in $L_1 \times L_2$ the relation $x \equiv y$ $(\Theta[S])$ holds. This does not hold trivially, for in general $x \vee s \geqq x \cup s$. But, from $(1, 0) \in L$ and from $x \cup (1, 0) \geqq x$ it follows that $x \leqq x \vee s \leqq x \cup (1, 0)$, and so $x \equiv x \cup (1, 0)$ $(\Theta[S])$, thus $x \equiv x \cup s$ $(\Theta[S])$ as we wished to prove.

Thus S is a standard ideal. $S \sim L_1$ is trivial, and $L/S \sim L_2$ is also immediate, thus the proof of Theorem 17 is completed.

It is clear from Theorem 6 that if the kernel were defined to be a neutral ideal, then we would not get the meet-sublattices, but the sublattices of $L_1 \times L_2$ as the solution of the extension problem.

A few words about the conditions: L_1 has a unit and L_2 has a zero element. The first hypothesis can easily be omitted, we have only to refer to Theorem 4, but obviously, without this condition the theorem would be more difficult.

The second condition is much more important. It assures that L_1 may be regarded as an ideal of $L_1 \times L_2$. Omitting this supposition, some new idea seems to be necessary to settle the question.

It seems that the standard ideals are the possible widest generalization of the class of neutral ideals, for which the Schreier problem has a solution. For instance, it is probable that there is no analogue of Theorem 17 for distributive ideals. To be precise we may say the following: it is natural to require in solving the Schreier problem that we get a finite general algebra as an extension of a finite general algebra by another finite one. This condition holds for groups, semigroups, rings and lattices (Theorem 17). It should not hold if we supposed the kernel to be only a distributive ideal. An example

for this is given by the lattice of Fig. 4, which is the Schreier extension of **2** by **2**, and the kernel of this extension is a distributive ideal.

PROBLEM 14. Try to omit the hypothesis that L_2 has a zero in Theorem 17.

CHAPTER V

CHARACTERIZATIONS OF NEUTRAL ELEMENTS IN MODULAR LATTICES

§ 1. Characterizations by distributive equalities

The notion of standard elements has been defined by (9) which is a distributive equality. Furthermore, the condition (δ) (i) of Theorem 1 is also a distributive equality. Thus we see that there is a close connection between distributive equalities and standard elements.

This connection is deeper in modular lattices as a consequence of Theorem IV of ORE which asserts that in modular lattices both (6) and (7) are capable of the definition of neutral elements.

First the question arises what the situation is with the distributive equality (8). Is it also capable of definition of the neutral elements in modular lattices? Further: one can imagine that there are in the class of modular lattices other equalities, capable of the characterization of the distributivity. Are all of these identities able to define the neutrality of an element?

In general, in lattice theory, distributive equality means one of the equalities (6), (7) and (8). The cause why these are called distributive equalities is given by the following theorem (DEDEKIND [7], MENGER [22]): let $f(x, y, z) = g(x, y, z)$ be any one of the equalities (6), (7), (8); then a distributive lattice may be defined as a lattice of type $f = g$. Basing upon this theorem, we give the definition of distributive equality as follows:

Let $\mathfrak{L}$ be a class of lattices. $f(x, y, z) = g(x, y, z)$ is a *distributive equality* of $\mathfrak{L}$ if and only if within $\mathfrak{L}$ the distributive lattices are just the lattices of type $f = g$.

We remark that f and g are elements of $FL_{\mathfrak{L}}(3)$, that is, the free lattice with three generators over $\mathfrak{L}$. ($FL_{\mathfrak{L}}(3)$ consists of all polynomials of three indeterminates and $f(x, y, z) = g(x, y, z)$ in $FL_{\mathfrak{L}}(3)$ if and only if $f(a, b, c) = g(a, b, c)$ for all $a, b, c \in L \in \mathfrak{L}$.)

We say that the distributive equality $f = g$ is equivalent to the distributive equality $f' = g'$ over $\mathfrak{L}$ if for all $a, b, c \in L \in \mathfrak{L}$, $f(a, b, c) = g(a, b, c)$ is equivalent to $f'(a, b, c) = g'(a, b, c)$. This amounts to the coincidence of the congruence relations $\Theta_{f,g}$ and $\Theta_{f',g'}$ of $FL_{\mathfrak{L}}(3)$.

Now, our aim is to determine all non-equivalent distributive equalities of the class of modular lattices, and to decide whether it is possible or not to define by any one of them the neutrality of an element.

We reach both of our aims by proving the following theorem:

THEOREM 18. *Let $f(x, y, z) = g(x, y, z)$ be a distributive equality of modular lattices. The elements a, b, c of the modular lattice L generate a distributive sublattice of L if and only if $f(a, b, c) = g(a, b, c)$.*

Before proving the theorem, let us draw some consequences of it:

COROLLARY 1. *Let $f(x, y, z) = g(x, y, z)$ be an arbitrary distributive equality of modular lattices. An element n of the modular lattice L is neutral if and only if $f(n, a, b) = g(n, a, b)$ for all $a, b \in L$.*

The proof is obvious, comparing the definition of neutrality with the assertion of Theorem 17.

COROLLARY 2. *In the class of modular lattices all distributive equalities are equivalent.*

PROOF. Let $f(x, y, z) = g(x, y, z)$ and $f'(x, y, z) = g'(x, y, z)$ be distributive equalities. If $f(a, b, c) = g(a, b, c)$ for $a, b, c \in L$, then by Theorem 17 the sublattice generated by a, b, c is distributive, thus $f'(a, b, c) = g'(a, b, c)$, and conversely. That means the equivalence of any two distributive equalities.

The proof of the theorem is based on the following lemma which seems to have an interest of his own.

LEMMA 16. *Let L be a modular but not distributive lattice generated by three elements. L has a homomorphism onto V, that is, onto the modular but not distributive lattice of five elements.*

PROOF. Let a, b, c be generators of L and p, q, r those of V. The correspondence $a \to p$, $b \to q$, $c \to r$ may be extended to a homomorphism of L onto V. It is rather easy to show that this correspondence is a homomorphism, but one can spare the trouble of this proof by a simple reference to a theorem of BIRKHOFF, which effectively describes the free modular lattice with three generators (see Fig. 15).

So far the lattice V has been used as a sublattice to characterize the non-distributivity of a modular lattice. Lemma 16 shows that it may be used also as a homomorphic image to characterize the non-distributivity in case L is generated by three elements. In general this is not true, as shown by the modular lattice of six elements, having four atoms. This lattice is simple, therefore it has no homomorphism onto V.

PROOF OF THEOREM 18. Suppose the theorem is not true, that is, there exists a modular lattice L, three elements a, b, c of L and a distributive

equality $f(x, y, z) = g(x, y, z)$ such that $f(a, b, c) = g(a, b, c)$ and the sublattice L_1 generated by a, b, c is not distributive. Then by Lemma 16 it has a homomorphism onto V such that the homomorphic images of a, b, c are p, q, r. Consequently, $f(p, q, r) = g(p, q, r)$. Let u, v, w be three elements of V distinct from p, q, r. Then these elements generate a distributive sublattice of V, thus $f(u, v, w) = g(u, v, w)$. We have shown that $f(u, v, w) = g(u, v, w)$ for all $u, v, w \in V$, that is, V is of type $f = g$. This is obviously a contradiction to the definition of distributive equalities. The proof is complete.

Finally, we make some general remarks. Let $\mathfrak{L}$ be a class of lattices, $f, g, f', g' \in FL_{\mathfrak{L}}(3)$. We say that $f = g$ is equivalent to $f' = g'$ if $\Theta_{f,g} = \Theta_{f',g'}$, although this is not the generally used definition. In [6], though not explicitly stated, another definition is used; this we shall now call the definition of weak equivalence. Denote by $\mathfrak{L}_{f=g}$ the class of lattices of type $f = g$ in the class $\mathfrak{L}$. We say that $f = g$ is weakly equivalent to $f' = g'$ if $\mathfrak{L}_{f=g} = \mathfrak{L}_{f'=g'}$.

An example of weakly equivalent but not equivalent equalities: $x \cup y \cup (x \cap y \cap z) = x \cup y \cup z$ and $x \cap y \cap (x \cup y \cup z) = x \cap y \cap z$. They are weakly equivalent, for they define the lattice of one element, but they are not equivalent, for in the lattice of Fig. 13 the elements a, b, c satisfy only one of them.

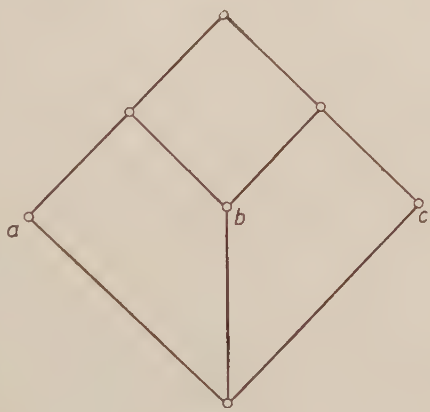

Fig. 13

It is trivial that per definitionem any two distributive equalities are weakly equivalent. Theorem 17 asserts that they are equivalent in modular lattices. The same assertion in the class of all lattices is not true, as it is shown again by the lattice of Fig. 13, where $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ but $a \cup (b \cap c) \neq (a \cup b) \cap (a \cup c)$.

Problem 15. Let degree $\mathfrak{m}$ (an infinite or finite cardinal) of non-distributivity of the modular lattice L be defined as the power of a subset H of L maximal with respect to the property that any three elements of H generate a non-distributive sublattice of L. Is $\mathfrak{m}$ an invariant of the lattice? Is it true that L has the degree of non-distributivity 3 if and only if it has a homomorphic image isomorphic to V? If the degree of non-distributivity of L is $\mathfrak{m}$, has L homomorphism onto a simple lattice, having the same degree of non-distributivity?

§ 2. Neutral elements as elements with unique relative complements

A theorem of J. VON NEUMANN (see [6]) asserts that an element n of a complemented modular lattice L is neutral if and only if its complement is unique. In modular lattices the same assertion does not hold in general; the element a of the lattice of Fig. 14 is uniquely complemented, but it is not neutral. This observation is due to HALL (see [6]). Applying twice Ex. 2 of p. 115 of [6] we get[8] that an element n of a complemented modular lattice L is uniquely complemented if and only if it is uniquely relatively complemented, that is, if it has just one relative complement in any interval containing n. In this way it is possible to generalize NEUMANN's theorem to arbitrary modular lattices.

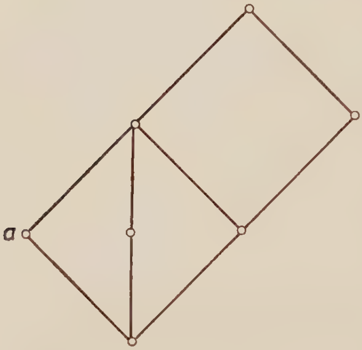

Fig. 14

THEOREM 19. *An element n of a modular lattice L is neutral if and only if it has at most one relative complement in any interval containing it.*

PROOF. If n is neutral, then by condition (ii) of Theorem III it obviously satisfies the stated condition.

Let n be an element of the modular lattice L, and x, y arbitrary elements of L. The free modular lattice FML 3), generated by n, x, y, according to a theorem of BIRKHOFF [6] is given in Fig. 15. If n satisfies the condition of Theorem 19, then $u = v$, for u and v are the relative complements of a in the interval $[a \cap u, a \cup u]$. It follows that the lattice generated by a, x and y must be a homomorphic image of $FML(3)(\Theta_{uv})$. But $FML(3)(\Theta_{uv})$ is distributive and it follows that n is neutral, as asserted.

COROLLARY 1. (Theorem of NEUMANN.) *In a complemented modular lattice an element n is neutral if and only if it has precisely one complement.*

Indeed, as we have remarked above, in complemented modular lattices an element n is uniquely relatively complemented if and only if it is uniquely complemented. Hence the corollary.

COROLLARY 2. *An element n of the modular lattice L is neutral if it is neutral in every interval $[n \cap x, n \cup x]$ $(x \in L)$.*

Corollary 2 is an immediate consequence of Theorem 19.

In weakly modular lattices Theorem 19 fails to be valid. In the lattice of Fig. 16 the element a is uniquely relatively complemented, but not neutral.

[8] See also [32].

Corollary 2 is obviously true in relatively complemented and in complemented lattices. For if L is relatively complemented and $n \in L$ is neutral in any $[n \cap x, n \cup x]$ $(x, y \in L)$, then let a be a relative complement of n in $[n \cap x \cap y, n \cup x \cup y]$. Since n is neutral in $[n \cap a, n \cup a]$, $\{n, x, y\}$ is a distributive lattice, and so n is neutral. But Corollary 2 is not true in general. Consider the lattice of Fig. 17. In this lattice a is not neutral, but it is in every interval of the form $[a \cap x, a \cup x]$.

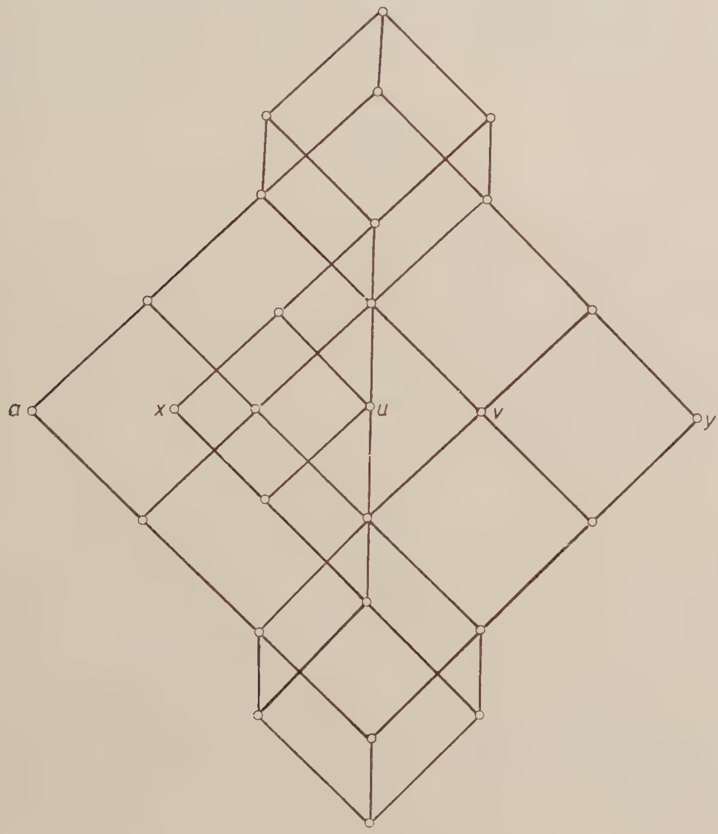

Fig. 15

PROBLEM 16. Is the assertion of Theorem 19 true in relatively complemented lattices?

PROBLEM 17. Is the assertion of Corollary 2 of Theorem 19 true in weakly modular lattices?

NOTE. The corresponding assertion for standard elements is not true. If $s \in L$ is locally standard (i. e. standard in every interval $[s \cap x, s \cup x]$) and

L is weakly modular, it does not follow that s is standard, as shown by the lattice of Fig. 18.

PROBLEM 18. Call an element s of L a *standard element of order two* if it is a standard element of a standard ideal of S. Do the standard elements of order two form a sublattice of L? Characterize the standard elements of order two. What can be said of the standard elements of higher order?

REMARK. The following theorem may be useful: if s is a standard element of order two of L, then there is a unique maximal standard ideal S such that s is standard in S.

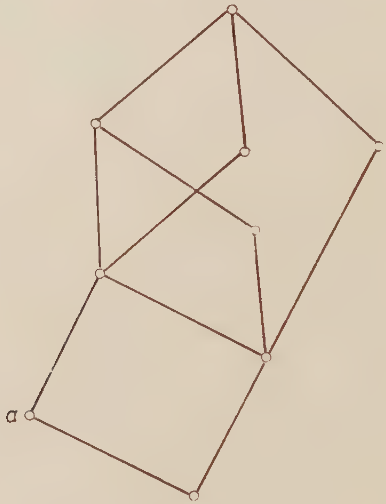

Fig. 16

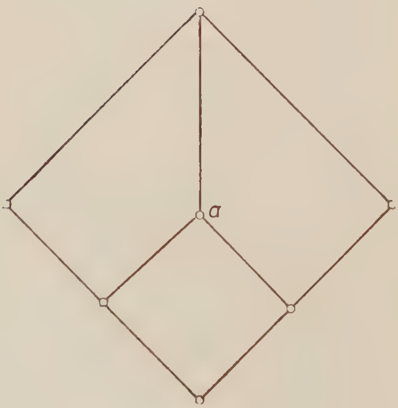

Fig. 17

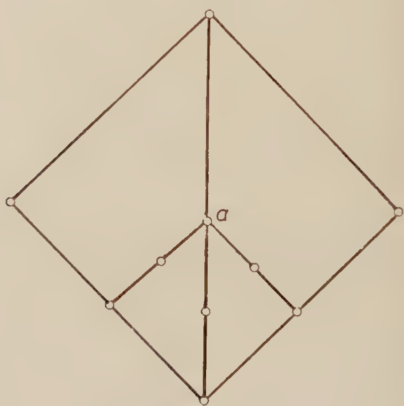

Fig. 18

CHAPTER VI

IDEALS SATISFYING THE FIRST ISOMORPHISM THEOREM

§ 1. The case of relatively complemented lattices with zero satisfying the ascending chain condition

In this section we want to prove the following theorem:

THEOREM 20. *Let L be a relatively complemented lattice with zero in which the ascending chain condition holds. An ideal I of the lattice L satisfies the first isomorphism theorem, i. e.*

$$I \cup K/I \simeq K/I \cap K$$

for all ideals K of L if and only if I is neutral.

PROOF. If I is neutral, then from Theorem 13 it follows that it satisfies the first isomorphism theorem. Now let us suppose that the ideal I of the relatively complemented lattice L with the ascending chain condition satisfies the first isomorphism theorem. By the structure theorem of DILWORTH, $L = L_1 \times \cdots \times L_k$ with simple lattices L_j, and consequently, $I = I_1 \times \cdots \times I_k$ with $I_j \subseteq L_j$ $(j = 1, \ldots, k)$. We prove that I satisfies the first isomorphism theorem if and only if every I_j satisfies in L_j the first isomorphism theorem. This is an immediate consequence of the following identity:

$$L/I \simeq L_1/I_1 \times \cdots \times L_k/I_k.$$

Hence we have reduced the question to the case of simple relatively complemented lattices. Instead of this we shall now consider a bit more general class of lattices, which will lead not only to the proof of Theorem 20, but at the same time to a generalization of it:

LEMMA 17. *Let L be a complemented simple lattice. No principal ideal of L except for* $(0]$ *and* $(1]$ *satisfies the first isomorphism theorem.*

PROOF. Let $a \in L$, and b the complement of a. Applying the first isomorphism theorem, we get

$$(a \cup b]/(a] \simeq (b]/(a \cap b].$$

The left member is isomorphic to the lattice of one element (except $a = 0$) and the right member is isomorphic to the principal ideal $(b]$. This is a contradiction, unless $a = 0$ or $b = 0$, as stated.

Lemma 17, compared to the arguments we have made above (using the structure theorem given in Corollary 4 of Theorem 11), leads to the following generalization of Theorem 20:

THEOREM 21. *Let L be a section complemented weakly modular lattice with the ascending chain condition. An ideal I of L satisfies the first isomorphism theorem if and only if it is neutral.*

COROLLARY. *Let I be a section complemented weakly modular lattice satisfying the ascending chain condition (for instance, a finite relatively complemented lattice) in which the first isomorphism theorem unrestrictedly holds. Then I is a finite Boolean algebra.*

In § 3 we shall show that the assertion of Theorem 20 holds in finite modular lattices as well. But it is not already true for finite weakly modular lattices. Fig. 17 shows a lattice L which is simple. The dual $\tilde{L}$ of L has an element $s \neq 0, 1$ which is locally standard[9] and thus it satisfies the first isomorphism theorem.

§ 2. A general theorem

In the following section we want to prove that the conclusion of Theorem 20 holds in modular lattices of locally finite length with zero. In this section we prove a general theorem which characterizes the ideals satisfying the first isomorphism theorem in certain classes of lattices which are a little more general than the class of finite lattices. The condition of this theorem is rather difficult, but starting from this, we shall be able to solve the problem in modular lattices of locally finite length with zero.

First we turn our attention to proving two lemmas of preliminary character.

LEMMA 18. *Let L be a lattice of finite length and let L satisfy the Jordan—Dedekind chain condition. Then $L(\Theta) \cong L$ $(\Theta \in \Theta(L))$ implies $\Theta = \omega$, that is, no proper homomorphic image of L is isomorphic to L.*

PROOF. Suppose $\Theta \neq \omega$ and $L(\Theta) \cong L$. Then there exist $a, b \in L$ such that $a > b$ and $a \equiv b$ (Θ). Let C be a maximal chain of length n such that $a, b \in C$. The image of C under Θ is a maximal chain of $L(\Theta)$, and its length is at most $n-1$, for the homomorphic images of a and b are the same. From the isomorphism $L \cong L(\Theta)$ it follows that the Jordan—Dedekind chain condition holds in $L(\Theta)$, and consequently the length of $L(\Theta)$ is at most $n-1$. Since the length of the original lattice is n, therefore this contradicts the isomorphism of L and $L(\Theta)$.

COROLLARY. *Let L be a lattice of finite length and suppose that every homomorphic image of L satisfies the Jordan—Dedekind chain condition. Then $\Theta, \Phi \in \Theta(L)$ and $\Theta \leq \Phi$, further $L(\Theta) \cong L(\Phi)$ imply $\Theta = \Phi$.*

[9] See the definition in the Note after Problem 17.

We remark that the conclusion of Lemma 18 does not hold if the Jordan—Dedekind chain condition is not presupposed. An example is given by Fig. 19. Let Θ be the congruence relation Θ_{ab}. Then $\Theta_{ab} > \omega$ and $L(\Theta_{ab}) \sim L$.

A generalization of the first isomorphism theorem is a homomorphism theorem which is always true:

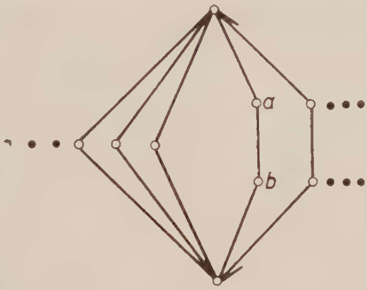

Fig. 19

Lemma 19. *Let I and K be arbitrary ideals of L. Then we have*

$$K \sim K/I \cap K \sim I \cup K/I.$$

Supplement 1. *A natural congruence relation Θ_1 of K under which $K(\Theta_1) \simeq K/I \cap K$ may be given in the following way: the elements a, b of K are congruent under Θ_1 if and only if there exist $u, v \in I \cap K$ and $a \cup b = x_0 \geqq x_1 \geqq \cdots \geqq x_n = a \cap b$ such that $\overline{u, v} \to \overline{x_{i-1}, x_i}$ $(i = 1, 2, \ldots, n)$ within K.*

Supplement 2. *A natural congruence relation Θ_2 of K under which $K(\Theta_2) \simeq I \cup K/I$ may be given as follows: the elements a, b of K are congruent under Θ_2 if and only if there exist $u, v \in I$ and $a \cup b = y_0 \geqq y_1 \geqq \cdots \geqq y_n = a \cap b$ such that $\overline{u, v} \to \overline{y_{i-1}, y_i}$ $(i = 1, 2, \ldots, n)$ within $I \cup K$.*

Without loss of generality we may suppose $L = I \cup K$. Consider the congruence relation $\Theta[I]$. This makes a partition of K into congruence classes and, obviously, this partition of K is compatible. Consider the congruence relation Θ^* of K which induces the same partition on K. We assert that

$$I \cup K/I \simeq K(\Theta^*).$$

From the first general isomorphism theorem we get that it is enough to prove that every congruence class of $I \cup K$ modulo $\Theta[I]$ contains an element from K. Indeed, let $x \in I \cup K$, then with suitable $y \in I$ and $z \in K$ we have $x \leqq y \cup z$. We put $t = x \cap y \cap z$ $(\in I)$. Then $y \equiv t$ $(\Theta[I])$, hence $y \cup z \equiv t \cup z = z$ $(\Theta[I])$, and so $x = x \cap (y \cup z) \equiv x \cap z$ $(\Theta[I])$. Thus $x \cup z \in K$ is congruent to z modulo $\Theta[I]$.

On the other hand, denote Θ_0 the congruence relation of the lattice K generated by $I \cap K$ (i. e. $\Theta_0 = \Theta[I \cap K]$ on the lattice K). Obviously, $a \equiv b$ (Θ_0) implies $a \equiv b\,(\Theta^*)$, that is, $\Theta_0 \leqq \Theta^*$. Thus $K(\Theta_0) \sim K(\Theta^*)$. We have already seen that $I \cup K/I \simeq K(\Theta^*)$, hence we have $K \sim K/K \cap I \sim I \cup K/I$ which was to be proved.

The assertion of the supplements is immediate if we compare the definitions of Θ^* and Θ_0 with that of Θ_2 and Θ_1 and with Theorem I and formula (3).

Now it will be easy to give the required characterization of the ideals satisfying the first isomorphism theorem.

THEOREM 22. *Suppose that every element a of L satisfies one of the following conditions:*

(a) $[0, a]$ *is a finite lattice;*

(b) *every homomorphic image of the lattice* $[0, a]$ *satisfies the Jordan—Dedekind chain condition.*

Then an ideal I of the lattice L satisfies the first isomorphism theorem with any principal ideal $K = (k]$ if and only if whenever the weak projectivity $u, v \rightarrow x, y$ holds within $I \cup K$, where $u, v \in I$ and $x, y \in K$, $x \succ y$, then with suitable elements w, z of $I \cap K$ the weak projectivity $\overline{w, z} \rightarrow \overline{x, y}$ holds within K.

PROOF. From the conditions it follows that if $K = (k]$ is a principal ideal, then for the lattice $[0, k]$ the conclusion of Lemma 18 is true. Indeed, if K satisfies condition (b) of Theorem 22, then the assertion follows from Lemma 18. If K satisfies condition (a) of Theorem 22, then it is a finite lattice. Let $\Theta, \Phi \in \Theta(K)$ and $\Theta < \Phi$. Then $K(\Phi)$ consists of fewer elements than $K(\Theta)$, thus $K(\Theta) \simeq K(\Phi)$ is impossible.

Now, in Lemma 19 we have seen that $I \cup K/I \sim K(\Theta^*)$ and $K/I \cap K \sim \sim K(\Theta_0)$, where the congruences Θ^* and Θ_0 of K have been defined in the proof of Lemma 10. Thus, if K satisfies the first isomorphism theorem, then from the previous section and from $\Theta_0 \leq \Theta^*$ it follows that $\Theta_0 = \Theta^*$. From conditions (a) and (b) we conclude that the lattice L is discrete, that is, any two elements $a > b$ of L may be connected by a finite maximal chain. It follows that two congruence relations, Θ_0 and Θ^*, are the same if and only if Θ_0 and Θ^* collapse the same prime intervals. But if a covers b, then in Supplement 1 of Lemma 19 we may take $n = 1$ and in Supplement 2 $m = 1$, and thus the coincidence of Θ_0 and Θ^* upon every prime interval is just assured by the condition of this theorem.

Conversely, if the conditions of Theorem 22 hold, then the congruence relations Θ_0 and Θ^* are the same, that is, $K(\Theta_0) \sim K(\Theta^*)$. Consequently, by Lemma 19, we get $I \cup K/I \sim K/I \cap K$, completing the proof of Theorem 22.

§ 3. Modular lattices of locally finite length with zero

The main result of this section is the following:

THEOREM 23. *Let L be a modular lattice of locally finite length with zero. An ideal I of L satisfies the first isomorphism theorem if and only if it is neutral.*

We prove a bit more than the assertion of the theorem, namely

COROLLARY. *Let L be a modular lattice with zero which is of locally finite length. The following four conditions on the ideal I of the lattice L are equivalent:*

(a) *I satisfies the first isomorphism theorem, that is, for an arbitrary ideal K of L*

$$I \cup K/I \simeq K/I \cap K;$$

(b) *I satisfies the first isomorphism theorem for an arbitrary principal ideal* $K=(k]$;

(c) *I is standard;*

(d) *I is neutral.*

We prepare the proof of this theorem with four lemmas. Among these the first is surely known, but we did not find in the literature. The most interesting of these lemmas is the third (Lemma 22) which gives the structure of an interesting free lattice.

LEMMA 20. *Let L be a locally finite modular lattice and I an ideal of L. A prime interval p of L collapses under* $\Theta[I]$ *if and only if it is projective to a prime interval q of I.*

PROOF. It is easy to derive this assertion from Theorem I and formula (3). A direct proof is the following: we define the relation Θ: $a \equiv b\ (\Theta)\ (a, b \in L)$ if and only if there exists a sequence of elements $a \cup b = y_0 \succ y_1 \succ \cdots \succ y_n = a \cap b$ such that each $[y_{i+1}, y_i]$ $(i=0, 1, \ldots, n-1)$ is projective to a prime interval of I. All the conditions of Lemma II are trivially satisfied for Θ, thus Θ is a congruence relation and $\Theta=\Theta[I]$ is also obvious. If $a \succ b$, then $n=1$, hence the assertion.

Let L be a modular lattice of locally finite length. Let us fix an ideal I and a prime interval p of L. By the previous lemma we can find prime intervals q in I such that $q \overset{n}{\longleftrightarrow} p$. Choose a q such that n be as small as possible. For the proof of Theorem 22 it will be useful to call this smallest n the order of p relative to I. $n=0$ means that $p \subseteq I$ and p is of infinite order if p does not collapse under $\Theta[I]$.

LEMMA 21. *Let L be a modular lattice of locally finite length. An ideal I of L is standard if and only if the orders of the prime intervals of L relative to I are* 0, 1 *or infinite.*

PROOF. If I is standard, then by condition (γ'') of Theorem 2 the assertion is obviously true.

Conversely, suppose the ideal I of L satisfies the condition. Let Θ be the relation defined in the proof of Lemma 20. If $a \equiv b\ (\Theta)$, then there exists

a sequence of elements $a \cup b = y_0 \succ y_1 \succ \cdots \succ y_n = a \cap b$ such that each $[y_{i+1}, y_i]$ is projective to a prime interval $[a_{i+1}, a_i]$ of I. Because all the $[y_{i+1}, y_i]$ are of the order 1 or 0 relative to I, it follows that we may suppose $y_{i+1} \cup a_i = y_i$. Let $x = \bigvee a_i$, then obviously $y_n \cup x = y_0$. This means that I satisfies condition (γ'') of Theorem 2 and, consequently, I is standard.

Corollary. *An ideal I of a modular lattice L of locally finite length is not standard if and only if there exists a prime interval p of L of order 2 relative to I.*

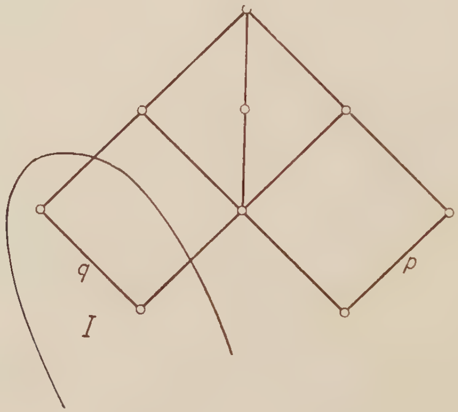

Fig. 20

The sublattice V (that is, the lattice of five elements which is modular but not distributive) is called *minimal* if its length in L is 2.

Now suppose that I is a non-standard ideal of the modular lattice L of locally finite length and consider a prime interval p of order 2, the existence of which is guaranteed by Lemma 21. It is possible that we can reach p from I in the way shown by Fig. 20. In this case the "turn" is through a minimal V. If this is the case, then we call the zero element of the minimal V a *turning element*. Consequently, if we can find to the ideal I a turning element, then I is surely not standard. The most important point of the proof of Theorem 22 is the converse of this statement. We cannot prove directly this assertion. First we have to find the most general situation which may occur in Fig. 20, that is, the corresponding free lattice.

Lemma 22. *Let L be a modular lattice of locally finite length. I is a non-standard ideal, p a prime interval of order 2 relative to I, q a prime interval of I and $q \overset{2}{\leftrightarrow} p$, namely, $\overline{b, a} \xrightarrow{1} \overline{f, e} \xrightarrow{1} \overline{d, c}$, $p = [b, a]$ and $q = [d, c]$.*

The free modular lattice generated by the elements a, b, c, d and e, f is the following:

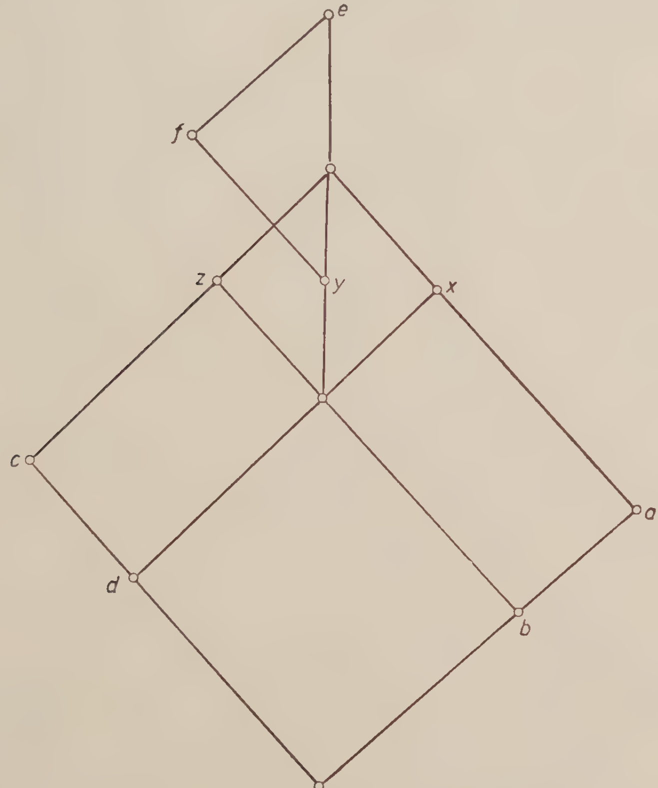

Fig. 21

REMARK. Simply to say, in Lemma 22 there is determined the free modular lattice generated by two covering pairs of elements. It is interesting the existence of this free lattice, for, in general, it is not allowed to prescribe covering relations in a free lattice.

PROOF. Consider the elements $x = a \cup d$, $y = f \cap (a \cup c)$, $z = b \cup c$, further the elements a, b, c, d, e, f, $b \cup d$, $a \cup c$, $b \cap d$. We prove that from the modularity and from the fact that the order of p relative to I is 2, finally from the covering relations it follows that these elements form a sublattice of L and all the joins and meets are the same as in Fig. 21 and these are consequences of the hypotheses.

First we show that x, y and z generate a minimal V, and

$$x \cup y = y \cup z = z \cup x = a \cup c, \quad x \cap y = y \cap z = z \cap x = b \cup d.$$

$x \cup z = a \cup c$ is clear from the definitions. We have $x \cup y = (a \cup d) \cup [f \cap (a \cup c)] =$ (from $a \cup d \leqq a \cup c$ and from the modularity) $= (a \cup d \cup f) \cap (a \cup c) =$ (because $a \cup f = e$) $= (d \cup e) \cap (a \cup c) = a \cup c$ for $c \leqq e$. We can get $y \cup z = a \cup c$ in a similar way.

From $x = a \cup d$ it follows that the interval $[b \cup d, x]$ is a transpose of $[b, a]$, and so it is of length 1 (we excluded the case $d \cup a = d \cup b$, for this implies $d \cap a > d \cap b$, thus $[d \cap b, d \cap a] \subseteq I$ is a transpose of $[b, a]$ the order of which relative to I is 1). Similarly, $[b \cup d, z]$ is also of length 1. Finally, $[b, a] \xrightarrow{1} [f, e] \xrightarrow{1} [y, a \cup c]$, and so $[y, a \cup c]$ is also a prime interval.

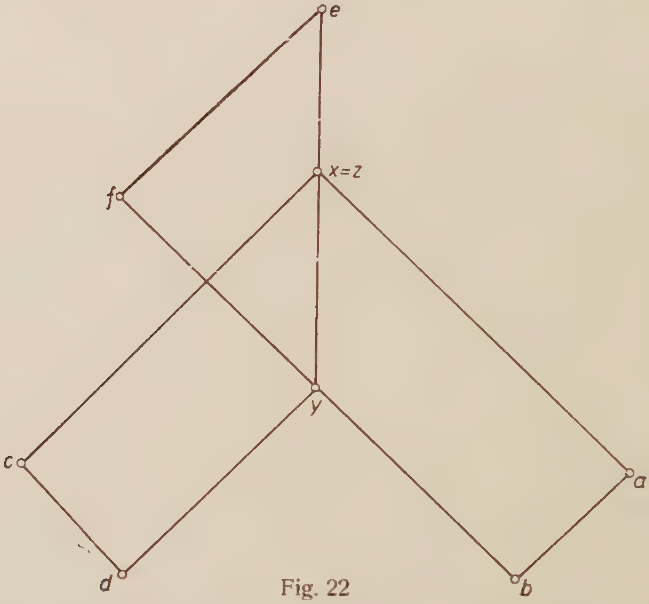

Fig. 22

We show that no two of x, y and z coincide. Suppose $x = z$. Because of $x \cup z = a \cup c$ we get $x = z = a \cup c$. Further, $[b \cup d, x]$ is a prime interval, and so $y = b \cup d$. In this case the diagram of the lattice (more precisely, a part of it) is shown by Fig. 22. We see that $a \cap c = a \cap d$ is impossible, for $a \cup c = a \cup d$ and $a \cup c = x$. Thus $[a \cap d, a \cap c]$ is a prime interval. Further, from $d \leqq y$ we get $a \cap d = a \cap y \cap d = b \cap d$. We prove that $[a \cap d, a \cap c]$ is a transpose of $[b, a]$. This will be a contradiction, for in this case the order of $[b, a]$ relative to I is 1 contrary to the hypotheses.

We have to prove that $b \cup (a \cap d) = b$ and $b \cup (a \cap c) = a$. We have $b \cup (a \cap d) =$ (from $a \cap d = b \cap d$) $= b$; further $b \cup (a \cap c) = (b \cup c) \cap a = z \cap a = a$.

The impossibility of $x = y$ is very easy to prove, for if $x = y$, then $f \geqq f \cap (a \cup c) = y = x = a \cup d \geqq a$, that is, $e = f \cup a = f$, a contradiction. We get a similar contradiction from $y = z$.

Of the remaining relations it is enough to prove $a \cap c = b \cap d$. Indeed, from $a \cap c = b \cap c$ (which is surely true, otherwise the order of $[b, a]$ relative to I were 1) and from $y \cap c = d$ we get $a \cap c = b \cap c = (y \cap b) \cap c = (y \cap c) \cap \cap b = d \cap b$. Thus the proof of Lemma 22 is completed.

Now we are able to prove the existence of turning elements.

LEMMA 23. *Let L be a modular lattice of locally finite length. An ideal I of L is non-standard if and only if there exists a turning element.*

PROOF. Using the notations of Lemma 21 and Lemma 22, consider the prime intervals $p = [b, a]$ and $q = [d, c]$ the existence of which is assured by Lemma 21. The sublattice of L generated by a, b, c, d and e, f is a homomorphic image of the free lattice of Lemma 22. Under this homomorphism, the minimal V of the free lattice does not collapse. (Indeed, if the minimal V collapsed, then both p and q would collapse.) Thus the minimal element of the minimal V may serve as a turning element.

We remark that the only congruences of the free lattice of Lemma 22, under which p does not collapse, are $\Theta_{fy}, \Theta_{xa}, \Theta_{cz}$ and their joins.

Now we are prepared for proving Theorem 23.

PROOF OF THEOREM 23. We prove the Corollary, for it is a stronger assertion than the theorem.

(c) implies (d) — this is stated in Lemma 10.

(d) implies (a) — this was proved in Theorem 13.

(a) implies (b) — this is trivial.

Thus we have to prove that (b) implies (c).

For this reason, let us suppose that L is a modular lattice with zero and of locally finite length, and I is an ideal satisfying

$$I \cup K/I \simeq K/K \cap I$$

for all $K = (k]$. If (b) does not imply (c), then I is not standard, that is, by Lemma 23 there exist turning elements in L. From the suppositions on L we obtain the existence of the dimension function $d(x)$. We denote from now on by u a turning element for which $d(u)$ is as small as possible. Finally, we denote by p and q the prime intervals (see Lemmas 22 and 23) from which the turning element u has been constructed.

Now we apply Theorem 21 to I and $(a]$. We may do so, for every interval $[0, a]$ of L satisfies the Jordan—Dedekind chain condition, and the same is true for any homomorphic image of $[0, a]$. $I \cup (a]$ contains the prime interval p, the order of which relative to I in $I \cup (a]$ is 2, for $I \cup (a]$ contains the whole minimal V because of $x \cup y \cup z = a \cup c \in (a] \cup I$. The order of p relative to $I \cap (a]$ in $(a]$ is at most 2, for if it were 1, then the order of p rela-

tive to I in $I \cup (a]$ would be also 1. Thus by Theorem 21 and Lemma 20 we can find a prime interval q_0 of $I \cap (a]$ and prime intervals $q_1, \ldots, q_n = p$ ($n \geqq 2$) of $(a]$ such that $q_0 \xrightarrow{1} q_1 \xrightarrow{1} \cdots \xrightarrow{1} q_n = p$. Let the intervals be chosen in such a way that n be the order of p relative to $I \cap (a]$ in $(a]$. Then q_2 is of order 2 relative to $I \cap (a]$ in $(a]$. It is trivial that the order of q_2 relative to I in $I \cup (a]$ is also 2, otherwise it would be 1, and this would imply the same in $(a]$. We may now apply Lemma 23 to q_0, q_2 and $I \cap (a]$, to conclude the existence of a turning element r. This turning element of $I \cap (a]$ in $(a]$ is a turning element of I in the whole lattice, too, for q_2 is of order 2 relative to I. But the minimal V, the zero of which is the turning element r, is included in $(a]$, therefore $d(r) < d(a) - 1$. On the other hand, $d(a) \leqq d(u) + 1$, thus $d(r) < d(u)$. We have found a turning element of lower dimension than u, a contradiction to the minimality of $d(u)$. This contradiction proves the Corollary of Theorem 22 and at the same time Theorem 22.

We should point out that as a consequence of Theorem 22 we get that every ideal satisfying the first isomorphism theorem is a homomorphism kernel in modular lattices with zero and of locally finite length. We have obtained the same conclusion in section complemented weakly modular lattices with ascending chain condition in § 1 of this chapter. Thus the following problem arises:

PROBLEM 19. Give classes of lattices in which every ideal satisfying the first isomorphism theorem is a homomorphism kernel. (Does the class of weakly modular lattices serve for this purpose?)

REMARK. In general it is not true, see, for instance, the ideal $(q]$ of the lattice U.

PROBLEM 20. Does there exist a modular lattice L and an ideal I of L such that I satisfies the first isomorphism theorem and despite this

a) I is not a homomorphism kernel, or

b) I is not a neutral ideal?

§ 4. A characterization of standard ideals by the first isomorphism theorem

In the Introduction we alluded to the fact that the notion of standard ideals is the best-possible one from the point of view of the first isomorphism theorem.

To formulate precisely what this means we need some notions.

Let $\mathfrak{A}$ be a class of ideals, i. e. if we are given a lattice L and an ideal I of L, then we are able to determine whether $I \in \mathfrak{A}$ or not. We say

that $\mathfrak{A}$ is of type $f_\alpha = g_\alpha$ $(\alpha \in A)$ if $I \in \mathfrak{A}$ is equivalent to the fact that I is of type $f_\alpha = g_\alpha$ $(\alpha \in A)$ in the sense of § 4 of Chapter III. We say that $\mathfrak{A}$ satisfies the first isomorphism theorem if from $I \in \mathfrak{A}$, I is an ideal of L it follows that I satisfies the first isomorphism theorem with any other ideal K of L. Finally, an ideal I of the lattice L is said to have the condition (**) if L_1/I is a sublattice of L/I under the natural mapping whenever $I \subseteq L_1 \subseteq L$, L_1 is a sublattice of L. Again, $\mathfrak{A}$ has property (**) if any of its ideals has it.

Only condition (**) needs a little explanation. It essentially requires that from the structure of L informations may be got about L/I.

For groups, (**) holds always (putting invariant subgroup instead of ideal and subgroup for sublattice).

Now we may state

THEOREM 24: *If the class $\mathfrak{A}$ of ideals*

1. *is of type $f_\alpha = g_\alpha$;*
2. *satisfies the first isomorphism theorem;*
3. *has the property* (**),

then $\mathfrak{A}$ contains only standard ideals.

The proof is easy, we have only to observe that it is an easy consequence of Theorem 10 that we may restrict ourselves to principal ideals. Now if $I = (d]$, then it may be easily proved that (*) of § 2 of Chapter III is equivalent to (**). As it was proved in § 3 of Chapter III, it follows that $\mathfrak{A}$ contains only distributive ideals. Now if d were distributive but not standard, then by Lemma 1 L would contain x, y with $x \geq y$, $d \cup x = d \cup y$, $d \cap x = d \cap y$. By 1, $(d] \in \mathfrak{A}$ in $\{d, x, y\}$ contradicting 2.

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ON A PROPERTY OF FAMILIES OF SETS

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1. Introduction. In this paper we are going to generalize a problem solved by MILLER in his paper [1] and prove several results concerning this new problem and some related questions. We mention here that some of our theorems (Theorems 8 and 10) have the interesting consequence that the topological product of $\aleph_k$ 1-compact spaces (Lindelöf spaces) is not necessarily k-compact for any finite k.[1]

DEF. (1.1) Let $\mathfrak{F}$ be a family of sets. $\mathfrak{F}$ is said by MILLER to possess property **B** if there exists a set B such that

$$F \cap B \neq 0 \qquad \text{for every} \qquad F \in \mathfrak{F},$$
$$F \nsubseteq B \qquad \text{for every} \qquad F \in \mathfrak{F}.$$

DEF. (1.2) Let $\mathfrak{F}$ be a family of sets. Let $p^*(\mathfrak{F})$ denote the least cardinal number p for which $\overline{\overline{F}} \leqq p$ for every $F \in \mathfrak{F}$. If $\overline{\overline{F}} \geqq p$ for every $F \in \mathfrak{F}$, we write $|\mathfrak{F}| \geqq p$. In what follows $p(\mathfrak{F}) = p$ denotes briefly that the family $\mathfrak{F}$ possesses the property

$$p^*(\mathfrak{F}) = p, \quad |\mathfrak{F}| \geqq p.$$

DEF. (1.3) Let $\mathfrak{F}$ be a family of sets and let $q \geqq 2$, $r \geqq 1$ be cardinal numbers. The family $\mathfrak{F}$ is said to possess property $\mathbf{C}(q, r)$ if $\bigcap_{F \in \mathfrak{F}'} F < r$ for every subfamily $\mathfrak{F}'$ of $\mathfrak{F}$, provided $\overline{\overline{\mathfrak{F}'}} \geqq q$.

NOTE. If for a family $\mathfrak{F}$ $|\mathfrak{F}| \geqq r$ and $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$, then $\mathfrak{F}$ consists of almost disjoint sets.

The result of MILLER which is our starting point can be stated as follows:

(1.4) Let p be an infinite cardinal number, n an integer ($n > 0$) and let $\mathfrak{F}$ be a family which possesses property $\mathbf{C}(p^+, n)$ such that $|\mathfrak{F}| \geqq p$. Then the family $\mathfrak{F}$ possesses property **B**.[2]

[1] In our example the spaces will be discrete ones. The generalized continuum hypothesis is used in the proof. As far as we know this result is new already for $k = 2$. This theorem should be compared with a theorem of J. Łos [3] (see Section 7).

[2] See [1], p. 35, Corollary.

To show that this result is best-possible MILLER proves the following:

(1. 5) There exists a family $\mathfrak{F}$ $(p(\mathfrak{F})=\aleph_0, \overline{\overline{\mathfrak{F}}}=2^{\aleph_0})$ which possesses property $\mathbf{C}(2, \aleph_0)$ and fails to possess property $\mathbf{B}$.[3]

However, one can ask what happens if $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_0)$ and $|\mathfrak{F}|$ is supposed to be greater than $\aleph_0$.

On the other hand, one can sharpen property $\mathbf{B}$ as follows:

DEF. (1. 6) Let $\mathfrak{F}$ be a family, s a cardinal number, $s \geqq 2$. $\mathfrak{F}$ is said to possess property $\mathbf{B}(s)$ if there exists a set B such that $F \cap B \neq 0$ and $\overline{\overline{F \cap B}} < s$ for every $F \in \mathfrak{F}$.

Our problems will be of the following kind. Let $\mathfrak{F}$ be a family of sets, and let m, p, q, r, s be cardinal numbers such that $\overline{\overline{\mathfrak{F}}}=m$, $p(\mathfrak{F})=p$ and suppose that $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$. Under what conditions for the cardinals m, p, q, r, s has $\mathfrak{F}$ to possess the properties $\mathbf{B}$ and $\mathbf{B}(s)$, respectively?

As the easy example (3. 3) will show, nothing can be said about property $\mathbf{B}(s)$ if $q>2$. The case $q=2$ contains the essential difficulty in the researches concerning the property $\mathbf{B}$ too.

The problem just stated is clearly a generalization of the problem treated in (1. 4) which is a corollary of [1], Theorem 1. We remark that it would be possible to generalize in a quite similar way the theorem itself not only its corollary, however, such a generalization does not seem to need new ideas and its formulation would be very complicated.

We restricted the formulation of the general problem with the assumption $p(\mathfrak{F})=p$ instead of MILLER's original assumption $|\mathfrak{F}| \geqq p$. This has no importance in the problems concerning property $\mathbf{B}$, however, in the problems for property $\mathbf{B}(s)$ it seems to be an essential restriction (see the remark at the end of Section 4).

2. Definitions. Notations. We use the usual notations of the set theory. We are going to list only those where there is a danger of misunderstanding.

In what follows $\mathfrak{F}, \mathfrak{G}, \ldots$ will denote families (sets of sets); capital letter will denote sets; $x, y, \ldots$ are the elements of the sets; m, t, p, q, r, s denote cardinals; $i, j, k, l, n, \ldots$ denote non-negative integers; $\alpha, \beta, \ldots$ denote ordinal numbers. Union and intersection of sets will be denoted by $\cup$ and $\cap$, respectively.

t^+ denotes the least cardinal greater than t (if t is finite, $t^+=t+1$). t^- is the immediate predecessor of the cardinal t if it exists, if not, then $t^-=t$. (If t is finite, $t^-=t-1$ for $t>0$ and $t^-=0$ for $t=0$.)

$\mathfrak{S}(S)$ denotes the set of all subsets of S.

[3] See [1], Theorem 3.

If $(x_\nu)_{\nu<\varphi}$ is an arbitrary sequence of type φ of not necessarily different elements x_ν, then $\{x_\nu\}_{\nu<\varphi}$ denotes the set of all x_ν's forthcoming in the sequence. This distinction will be sometimes omitted if there is no danger of misunderstanding.

Let $\varphi(x)$ be an arbitrary property of the elements of a set H. The set of all $x \in H$ which satisfy $\varphi(x)$ will be denoted by $\{x:\varphi(x)\}$. (We are going to use the logical signs $\wedge$ (and), $\vee$ (or) in the formulation of these formulas.)

The sets $\{X:X\subseteq S \wedge \overline{\overline{X}}=t\}$, $\{X:X\subseteq S \wedge \overline{\overline{X}}<t\}$ will be denoted by $[S]^t$, $[S]^{<t}$, respectively.

For an arbitrary family $\mathfrak{F}$ the set $\bigcup_{F\in\mathfrak{F}} F$ is denoted by $(\mathfrak{F})$. Other special notations concerning families will be introduced later.

For the study of the problem stated in the introduction we introduce the following symbols:

Def. (2.1) $\mathbf{M}(m,p,q,r)\rightarrow\mathbf{B}$ indicates the statement that every family $\mathfrak{F}$ which possesses property $\mathbf{C}(q,r)$ possesses property $\mathbf{B}$, provided $p(\mathfrak{F})=p$ and $\overline{\overline{\mathfrak{F}}}=m$.

Def. (2.2) $\mathbf{M}(m,p,r)\rightarrow\mathbf{B}(s)$ indicates the statement that every family $\mathfrak{F}$ which possesses property $\mathbf{C}(2,r)$ possesses property $\mathbf{B}(s)$, provided $p(\mathfrak{F})=p$ and $\overline{\overline{\mathfrak{F}}}=m$. $\mathbf{M}(m,p,q,r)\not\rightarrow\mathbf{B}$ and $\mathbf{M}(m,p,r)\not\rightarrow\mathbf{B}(s)$ denote the negations of the corresponding statements, respectively.

To exclude trivial exceptions here we assume once for all $m>0$, $p>0$, $q>1$, $r>0$, $s>1$.

We call briefly the symbols now introduced symbol-I and symbol-II, respectively.

The proof of some of our theorems makes use of the generalized continuum hypothesis or of the so-called measure hypothesis stated in [5]. These hypotheses will be cited as hypotheses (*), (**) and the corresponding theorems will be denoted by the same signs, respectively.

3. Preliminaries. A short summary of the content of the following sections. We briefly say that one of the symbols is monotone increasing (decreasing) in one of its variables, e. g. in m, if the fact that it is true for $m,p,q,r,(s)$ implies that it is true for $m',p,q,r,(s)$ for $m'\geqq m$ $(m'\leqq m)$, respectively. The following monotonicity properties are immediate consequences of the definitions (2.1) and (2.2):

(3.1) Both symbols are decreasing in m and r. Symbol-I is decreasing in q. Symbol-I is increasing in p, symbol-II is increasing in s.

We call attention that symbol-II is not increasing in p (see the end of Section 4).

It is also obvious that

(3. 2) $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ implies $\mathbf{M}(m, p, 2, r) \to \mathbf{B}$ if $s \leqq p$. (For $s > p$ symbol-II is trivially true.)

Now we prove:

(3. 3) *Let* $p \geqq \aleph_0$, $s \leqq p$ *be cardinal numbers. There exists a family* $\mathfrak{F}$ *such that* $F = p$, $p(\mathfrak{F}) = p$, $\mathfrak{F}$ *possesses property* $\mathbf{C}(3, 1)$ *and it does not possess property* $\mathbf{B}(s)$.

Proof. Let S be a set of power p. Let $\mathfrak{F}'$ be a system of subsets of S such that $\mathfrak{F}' = p$, $F_1 \cap F_2 = 0$ for $F_1, F_2 \in \mathfrak{F}'$, $F_1 \neq F_2$ and $F = p$ for every $F \in \mathfrak{F}'$. Put $\mathfrak{F} = \{S\} \cup \mathfrak{F}'$. It is obvious that $\mathfrak{F}$ satisfies the requirements of (3. 3) for $s = p$.

(3. 3) shows that in the investigations concerning property $\mathbf{B}(s)$ the assumption $q \leqq 2$, i. e. $q = 2$ is essential.

Miller's result (1. 4) can be stated as follows:

Theorem 1. *Suppose* $p \geqq \aleph_0$, $q \leqq p^+$. *Then for every* m *and for every* $r < \aleph_0$

$$\mathbf{M}(m, p, q, r) \to \mathbf{B}.$$

Miller's theorem can be considered as a generalization of Bernstein's theorem which states that if p is infinite, then every family $\mathfrak{F}$ ($\mathfrak{F} = p$, $p(\mathfrak{F}) = p$) (without any further assumption for property $\mathbf{C}(q, r)$) possesses property $\mathbf{B}$,[4] i. e.:

Theorem 2. $\mathbf{M}(p, p, q, r) \to \mathbf{B}$ *if* $p \geqq \aleph_0$ *for every* q *and* r.

Miller's counterexample (1. 5) can be stated generally as follows:

Theorem 3. $\mathbf{M}(2^p, p, 2, p) \not\to \mathbf{B}$ *if* p *is infinite.*

Theorem 3 can be proved quite similarly as its special case for $p = \aleph_0$ cited in (1. 5) and therefore we omit the proof.

Theorem 2 shows that in the investigations concerning property $\mathbf{B}$ we may always suppose that $m \geqq p$, and Theorem 3 shows that if $m > p$, then to obtain positive results we have to suppose $r < p$.

We mention that without using (*) we can not decide the following

Problem 1. $\mathbf{M}(\aleph_1, \aleph_0, 2, \aleph_0) \not\to \mathbf{B}$?

We can not prove without (*) that every $\mathfrak{F}$ ($p(\mathfrak{F}) = \aleph_0$, $\mathfrak{F} < 2^{\aleph_0}$) possesses property $\mathbf{B}$.

[4] See [4].

It is obvious that the property $\mathbf{C}(q', r)$ is weaker than the property $\mathbf{C}(q, r)$, provided $q' > q$.

Now we prove:

(3.4) $$\mathbf{M}(2^p, p, q, 1) \nrightarrow \mathbf{B} \quad if \quad q > 2^p, \quad p \geqq \aleph_0.$$

PROOF. Put $\mathfrak{F} = [S]^p$ where S is a set of power p. It is obvious that $\bar{\mathfrak{F}} = 2^p$, $p(\mathfrak{F}) = p$, $\mathfrak{F}$ possesses property $\mathbf{C}(q, 1)$, but it does not possess property $\mathbf{B}$.

(3.4) shows that we have to suppose $q \leqq 2^p$, and since here we do not want to discuss the difficulties caused by the continuum problem, we are going to suppose $q \leqq p^+$.

It results that the best-possible generalization of Theorem 1 would be the following:

(o) $\mathbf{M}(m, p, q, r) \rightarrow \mathbf{B}$ *for every* m, *provided* $q \leqq p^+$, $p \geqq \aleph_0$, $r < p$.

We can prove this only with the stronger assumption $r^+ < p$ (see Theorem 4) or with the restriction that m is not too large (see Theorem 5). Both proofs use (*).

The simplest unsolved problem here is

PROBLEM 2. $\mathbf{M}(\aleph_{\omega+1}, \aleph_1, 2, \aleph_0) \rightarrow \mathbf{B}$?

As to the symbol-II the problems are more ramified. First we have to discuss the case $m \leqq p$ which leads to some interesting result too. This will be done in Section 4. (3.2) shows that we have to suppose $s \leqq p$. In Section 4 we are going to prove that at least in the case $p \geqq \aleph_0$, $m \geqq p$ we may suppose $r^+ \leqq s$.

So the best-possible refinement of the conjecture (o) would be the following:

(oo) $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(r^+)$ *for every* $m \geqq p \geqq \aleph_0$, *provided* $r < p$.

Now we have to distinguish two cases:

(i) If r is finite, then (oo) is false. However, it is always true for $\aleph_0$ instead of r^+ and using (*) corresponding to every m, p and r we can determine the least s (eventually finite) for which $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ is true. These results will be proved in Section 7. As a consequence of these results we prove the topological theorem mentioned in the introduction (in Section 8). There we state many conjectures which all would have been consequences of 2-compactness of the topological product of $\aleph_2$ Lindelöf spaces — now disproved — and which we can not disprove with our method.

(ii) If r is infinite, (oo) is very likely true, however, we can prove it — using (*) — only with similar restrictions as in the case of symbol-I,

namely we can prove that

$$\mathbf{M}(m, p, r) \to \mathbf{B}(r^+) \quad \text{for} \quad m \geqq p > r > \aleph_0,$$

provided m is not too large (see Theorem 7), and we can prove that

$$\mathbf{M}(m, p, r) \to \mathbf{B}(r^{++}) \quad \text{for every} \quad m \geqq p > r^+ > \aleph_0$$

(see Theorem 6).

The simplest unsolved problems here are

PROBLEM 3.

a) $\mathbf{M}(\aleph_{\omega+1}, \aleph_{\omega+1}, \aleph_0) \to \mathbf{B}(\aleph_1)$?

b) $\mathbf{M}(\aleph_{\omega+1}, \aleph_1, \aleph_0) \to \mathbf{B}(\aleph_1)$?

c) $\mathbf{M}(\aleph_{\omega+1}, \aleph_2, \aleph_1) \to \mathbf{B}(\aleph_2)$?

The results on (o) and (oo) will be proved in Section 6. All the positive results concerning the case $m > p$ will be proved with the method of MILLER's theorem, and the proof runs always by induction on m. That is why we need a generalization of the induction process used in [1]. This will be done in Section 5 and as a corollary of it we obtain all the positive theorems (Theorems 4—9) already mentioned.

In Section 9 we deal with the case of finite sets ($p < \aleph_0$) and with some questions related to property **B**.

4. The symbol-II in the cases $m \leqq p$ ($p \geqq \aleph_0$). The following theorems of A. TARSKI will play an important role in our investigations:

(*) LEMMA 1. *Let S be a set, $\mathfrak{F}$ a family such that $(\mathfrak{F}) \subseteq S$, $S = \aleph_\alpha$, $|\mathfrak{F}| \geqq \aleph_\beta$. Then*

a) $\mathfrak{F} < \aleph_\alpha$, *provided $\mathfrak{F}$ possesses property* $\mathbf{C}(\aleph_{\alpha+1}, \aleph_\beta)$ *and* $cf(\alpha) \neq cf(\beta)$.

b) $\mathfrak{F} \leqq \aleph_\alpha$, *provided $\mathfrak{F}$ possesses property* $\mathbf{C}(\aleph_{\alpha+1}, r)$ *for an* $r < \aleph_\beta$.[5]

Note that in TARSKI's paper the theorems are proved under the stronger conditions that $\mathfrak{F}$ possesses the properties $\mathbf{C}(2, \aleph_\beta)$ and $\mathbf{C}(2, r)$, respectively, however, the proofs can be carried out in the same way for our case too.

LEMMA 2. *Let S be a set, $\mathfrak{F}$ a family such that $(\mathfrak{F}) \subseteq S$, $S = \aleph_\alpha$, $|\mathfrak{F}| \geqq r$ where r is finite. Then $\mathfrak{F} \leqq \aleph_\alpha$, provided $\mathfrak{F}$ possesses property* $\mathbf{C}(\aleph_{\alpha+1}, r)$.

Lemma 2 is a corollary of the fact that $[S]^r = \aleph_\alpha$ for every finite r. Note that the proof of Lemma 2 does not make use of (*).

(4. 1) $\mathbf{M}(m, p, r) \to \mathbf{B}(m^+)$ *for every* m, p, r.

$\mathbf{M}(m, p, r) \not\to \mathbf{B}(m)$ *if* $r > p$ *and* $p \geqq \aleph_0$ $(\aleph_0 \leqq m \leqq p)$.

[5] See [2], Theorem 5, I, p. 211 and Corollary 6, p. 213 for a) and b), respectively.

PROOF. The first statement is trivial, the second is to be seen quite similarly to (3. 3).

REMARK. *If* m *is finite,* $m \geqq 2$, *then* $\mathbf{M}(m, p, r) \to \mathbf{B}(m)$ *and* $\mathbf{M}(m, p, r) \not\to \mathbf{B}(m-1)$ *is true under the same conditions for* p, r *respectively.*

(4. 1) shows that we may always suppose $r \leqq p$.

(4. 2) $\mathbf{M}(m, \aleph_\alpha, r) \to \mathbf{B}(2)$ *for every* α *if* $m < \aleph_\alpha$ *and* $r < \aleph_\alpha$. *If* $r = \aleph_\alpha$, *then the same is true for* $m < \aleph_{cf(\alpha)}$.

PROOF. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F}) = \aleph_\alpha,\ \overline{\overline{\mathfrak{F}}} = m)$ which possesses property $\mathbf{C}(2, r)$. It is obvious that the set $F - \bigcup_{F' \in \mathfrak{F},\ F' \neq F} F'$ is of power $\aleph_\alpha$ and so it is non-empty for an arbitrary $F \in \mathfrak{F}$. Let x_F be an element of this set and put $B = \{x_F\}_{F \in \mathfrak{F}}$. We have $\overline{\overline{B \cap F}} = 1$ for every $F \in \mathfrak{F}$, hence $\mathfrak{F}$ possesses property $\mathbf{B}(2)$.

(4. 3) $\mathbf{M}(\aleph_{cf(\alpha)}, \aleph_\alpha, \aleph_\alpha) \to \mathbf{B}(\aleph_{cf(\alpha)})$ *for every* α.

PROOF. Let $\mathfrak{F}$ be a family such that $p(\mathfrak{F}) = \aleph_\alpha$, $\overline{\overline{\mathfrak{F}}} = \aleph_{cf(\alpha)}$, and suppose that $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$. Let $\mathfrak{F} = \{F_\nu\}_{\nu < \omega_{cf(\alpha)}}$ be a well-ordering of $\mathfrak{F}$.

The set $F_\nu - \bigcup_{\mu < \nu} F_\mu$ is of power $\aleph_\alpha$ for every $\nu < \omega_{cf(\alpha)}$. Let x_ν be an element of it.

Put $B = \{x_\nu\}_{\nu < \omega_{cf(\alpha)}}$. It is obvious that $B \cap F_\nu \neq 0$ for every $\nu < \omega_{cf(\alpha)}$ and $\overline{\overline{B \cap F_\nu}} < \aleph_{cf(\alpha)}$ for every $\nu < \omega_{cf(\alpha)}$, since if $\nu' > \nu$, then $x_{\nu'} \notin F_\nu$. It follows that $\mathfrak{F}$ possesses property $\mathbf{B}(\aleph_{cf(\alpha)})$.

Now we show that (4. 2) is best-possible in "s", i. e.

(4. 4) $\mathbf{M}(\aleph_{cf(\alpha)}, \aleph_\alpha, \aleph_\alpha) \not\to \mathbf{B}(s)$ *if* $s < \aleph_{cf(\alpha)}$.

PROOF. We are going to suppose that α is of the second kind. If α is of the first kind, the statement can be proved quite similarly. Let S be a set of power $\aleph_\alpha$ and let $S = \{x_\beta\}_{\beta < \omega_\alpha}$ be a well-ordering of type ω_α of S. Let $\{\alpha_\nu\}_{\nu < \omega_{cf(\alpha)}}$ be a monotone increasing sequence of type $\omega_{cf(\alpha)}$ of ordinal numbers less than α cofinal with α. Put $S_\nu = \{x_\beta\}_{\beta < \omega_{\alpha_\nu}}$ for every $\nu < \omega_{cf(\alpha)}$. Obviously, one can define the sequences $\{F^1_\nu\}_{\nu < \omega_{cf(\alpha)}}$, $\{F^2_\nu\}_{\nu < \omega_{cf(\alpha)}}$ of type $\omega_{cf(\alpha)}$ of subsets of S in such a way that — if we put $\mathfrak{F} = \{F^1_\nu\}_{\nu < \omega_{cf(\alpha)}} \cup \{F^2_\nu\}_{\nu < \omega_{cf(\alpha)}}$, then $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$ — and that the following conditions hold:

(1) $\overline{\overline{F^1_\nu}} = \overline{\overline{F^2_\nu}} = \aleph_\alpha$ for $\nu < \omega_{cf(\alpha)}$,

(2) $S_\nu \subseteq F^1_\nu$ for $\nu < \omega_{cf(\alpha)}$,

(3) $F^2_\nu \cap F^2_{\nu'} = 0$ for $\nu \neq \nu'$, $\nu, \nu' < \omega_{cf(\alpha)}$.

Then $p(\mathfrak{F}) = \aleph_\alpha$ by (1) and $\overline{\overline{\mathfrak{F}}} = \aleph_{cf(\alpha)}$. Suppose that the set B intersects every set F of the family $\mathfrak{F}$. Then $\overline{\overline{B}} \geqq \aleph_{cf(\alpha)}$ by (3), hence if $s < \aleph_{cf(\alpha)}$,

there is a $B' \subseteq B$ such that $\overline{\overline{B'}} = s$ and there is a $\nu_0 < \omega_{cf(\alpha)}$ such that $B' \subseteq S_{\nu_0}$. But this means by (2) that $\overline{\overline{B \cap F^1_{\nu_0}}} \geqq s$ and thus $\mathfrak{F}$ does not possess property $\mathbf{B}(s)$.

It results from (4. 1), (4. 2), (4. 3) and (4. 4) that to complete the discussion of the case $m \leqq p$ $(p \geqq \aleph_0)$ we have to determine the value of symbol-II in the following cases:

A) $m = p$, $r < p$ $(p \geqq \aleph_0)$,

B) $m = \aleph_\beta$, $p = \aleph_\alpha$, $r = \aleph_\alpha$ where $cf(\alpha) < \beta \leqq \alpha$.

To obtain complete results we have to assume (*) in both cases. In the case A) there remains an unsolved problem even if we assume (*).

First we prove the following negative result concerning A):

(*) (4. 5) $\mathbf{M}(\aleph_\alpha, \aleph_\alpha, r) \not\rightarrow \mathbf{B}(r)$ *if* $r < \aleph_\alpha$.

(If r is finite, the assumption (*) can be omitted.)

PROOF. Let $\mathfrak{F}_1, \mathfrak{F}_2$ be families satisfying the following conditions:

(4. 5. 1) $$\overline{\overline{\mathfrak{F}_1}} = r^+, \quad \overline{\overline{\mathfrak{F}_2}} = \aleph_\alpha,$$

(4. 5. 2) $$p(\mathfrak{F}_1 \cup \mathfrak{F}_2) = \aleph_\alpha,$$

(4. 5. 3) $$F \cap F' = 0 \quad \text{for every pair } F, F' \in \mathfrak{F}_1 \cup \mathfrak{F}_2, \ F \neq F'.$$

Put $(\mathfrak{F}_1) = S_1$. By Zorn's lemma there exists a *maximal* system $\mathfrak{S}$ of subsets of S_1 satisfying the following conditions:

(4. 5. 4) $$\overline{\overline{X \cap F}} = 1 \quad \text{for every} \quad X \in \mathfrak{S}, F \in \mathfrak{F}_1,$$
$$\overline{\overline{X \cap Y}} < r \quad \text{for every pair} \quad X, Y \in \mathfrak{S}, \ X \neq Y.$$

From (4. 5. 1) and (4. 5. 4) we get

(4. 5. 5) $$\overline{\overline{X}} = r^+ \quad \text{for every} \quad X \in \mathfrak{S}$$

and using the maximality of $\mathfrak{S}$ we obtain:

(4. 5. 6) If the set B' intersects every set F of $\mathfrak{F}_1$, then there exists an element X_0 of $\mathfrak{S}$ such that $\overline{\overline{B' \cap X_0}} \geqq r$.

On the other hand, using Lemmas 1 and 2 we get from (4. 5. 1), (4. 5. 2) and (4. 5. 4) that

(4. 5. 7) $$\overline{\overline{\mathfrak{S}}} = \aleph_\alpha.$$

It follows that there exists a one-to-one mapping $h(X)$ which maps $\mathfrak{S}$ onto $\mathfrak{F}_2$. Put $\mathfrak{F}_2^* = \{h(X) \cup X\}_{X \in \mathfrak{S}}$ and define $\mathfrak{F}$ as follows:

$$\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2^*.$$

Since $r^+ \leqq \aleph_\alpha$ by the assumption, by (4. 5. 2) and (4. 5. 5) we have $p(\mathfrak{F}) = \aleph_\alpha$. By (4. 5. 3) and (4. 5. 4) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$ and by (4. 5. 1) and (4. 5. 7) $\overline{\overline{\mathfrak{F}}} = \aleph_\alpha$.

We have to prove that $\mathfrak{F}$ does not possess property $\mathbf{B}(r)$. But if the set B intersects every set F of $\mathfrak{F}$, then it has a subset B' satisfying the condition of (4.5.6), hence $\overline{B \cap X_0} \geqq r$ for an $X_0 \in \mathfrak{S}$ and therefore $\overline{B \cap F_0} \geqq r$ for $F_0 = h(X_0) \cup X_0$, hence for an $F_0 \in \mathfrak{F}$. Thus $\mathfrak{F}$ does not possess property $\mathbf{B}(r)$.

REMARK. We have proved the following somewhat more general statement: *The family $\mathfrak{F}$ constructed above is such that each set which intersects every element of $\mathfrak{F}_1$ has to intersect an element of $\mathfrak{F}_2^*$ in at least r points.*

Now we need some preliminary definitions.

DEF. (4.6) Let $\mathfrak{F}$ be an arbitrary family and let S be a set such that $(\mathfrak{F}) \subseteq S$. For an arbitrary subset X of S and for an arbitrary cardinal number t we define the subfamily $\mathfrak{G}(X, t, \mathfrak{F})$ as follows:

$$\mathfrak{G}(X, t, \mathfrak{F}) = \{F : F \in \mathfrak{F} \wedge \overline{\overline{F \cap X}} \geqq t\}.$$

DEF. (4.7) Let $\mathfrak{F}$ and S have the same meaning as in (4.6). For an arbitrary $X \subseteq S$ we define the family $\mathfrak{F}|X$ as follows:

$$\mathfrak{F}|X = \{F \cap X\}_{F \in \mathfrak{F}}.$$

(Note that $\mathfrak{F}|X$ is not necessarily a subfamily of $\mathfrak{F}$.)

The following assertions are immediate consequences of the above definitions.

(4.8.1) *Let q, r be arbitrary. The families $\mathfrak{G}(X, t, \mathfrak{F})$ and $\mathfrak{F}|X$ possess property $\mathbf{C}(q, r)$, provided the same holds for $\mathfrak{F}$.*

(4.8.2) $|\mathfrak{G}(X, t, \mathfrak{F})|X| \geqq t$.

(4.8.3) *If the family $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$ and $t \geqq r$, then*

$$\overline{\overline{\mathfrak{G}(X, t, \mathfrak{F})}} \leqq q^{-} \cdot \overline{\overline{\mathfrak{G}(X, t, \mathfrak{F})|X}}.$$

Now we prove the following positive theorem concerning A):

(*) (4.9) *Suppose $r < \aleph_\alpha$. Then $\mathbf{M}(\aleph_\alpha, \aleph_\alpha, r) \to \mathbf{B}(r^+)$, provided the following condition does not hold:*

(v) *There exist ordinal numbers β, γ such that $\alpha = \beta + 1$, $r = \aleph_\gamma$, $cf(\beta) = cf(\gamma)$ and $\gamma < \beta$.*

(If r is finite, the assumption (*) can be omitted.)

PROOF. Let $\mathfrak{F}$ be a family ($p(\mathfrak{F}) = \aleph_\alpha$, $\overline{\overline{\mathfrak{F}}} = \aleph_\alpha$) which possesses property $\mathbf{C}(2, r)$. Put $S = (\mathfrak{F})$. Then $\overline{\overline{S}} = \aleph_\alpha$. Let $S = \{x_\nu\}_{\nu < \omega_\alpha}$ and $\mathfrak{F} = \{F_\mu\}_{\mu < \omega_\alpha}$ be well-orderings of type ω_α of the set S and of the family $\mathfrak{F}$, respectively. We may suppose that $r^+ < \aleph_\alpha$, for if not, then $r^+ = \aleph_\alpha$ is regular and the

theorem follows from (4. 3), since the symbol-II is decreasing in r by (3. 1). Now we define a subsequence $\{x_{\nu_\varrho}\}_{\varrho<\omega_\alpha}$ of S by induction on ϱ as follows:

Let x_{ν_0} be an arbitrary element of F_0 and put $F_0 = F_{\mu_0}$. Suppose that the elements x_{ν_σ} are already defined for every $\sigma<\varrho$, for a $\varrho<\omega_\alpha$. Put

(4.9.1) $$S_\varrho = \{x_{\nu_\sigma}\}_{\sigma<\varrho}, \quad \mathcal{G}_\varrho = \mathcal{G}(S_\varrho, r, \mathcal{F}).$$

It is obvious that $\overline{\overline{S_\varrho}} \leqq \bar{\varrho} < \aleph_\alpha$ and by (4.8.3) $\overline{\overline{\mathcal{G}_\varrho}} \leqq \overline{\overline{\mathcal{G}_\varrho | S_\varrho}}$.
But $\mathcal{G}_\varrho | S_\varrho \subseteq \mathcal{S}(S_\varrho)$, thus using (*) we get

(4.9.2) $$\overline{\overline{\mathcal{G}_\varrho}} < \aleph_\alpha \quad \text{except if} \quad \overline{\overline{S_\varrho}}^{+} = \bar{\varrho}^{+} = \aleph_\alpha.$$

Put $\bar{\varrho} = \aleph_\beta$ and suppose $\beta+1=\alpha$. Then $\overline{\overline{S_\varrho}} = \aleph_\beta$, $|\mathcal{G}_\varrho | S_\varrho| \geqq r$ (by 4.8.2). Using Lemmas 1 and 2 for the family $\mathcal{G}_\varrho | S_\varrho$ we get

(4.9.3) $$\overline{\overline{\mathcal{G}_\varrho}} \leqq \overline{\overline{\mathcal{G}_\varrho | S_\varrho}} \leqq \aleph_\beta < \aleph_\alpha \quad \text{except if} \quad r = \aleph_\gamma,\ cf(\beta) = cf(\gamma).$$

Thus it results from (4.9.2) and (4.9.3) and from the assumption that (v) does not hold that $\overline{\overline{\mathcal{G}_\varrho}} < \aleph_\alpha$. Let μ_ϱ be the least ordinal number μ for which

$$S_\varrho \cap F_\mu = 0.^{6}$$

It is obvious that $F_{\mu_\varrho} \notin \mathcal{G}_\varrho$, and so the set

$$F_{\mu_\varrho} - (\mathcal{G}_\varrho) = F_{\mu_\varrho} - \bigcup_{F_\mu \in \mathcal{G}_\varrho} (F_{\mu_\varrho} \cap F_\mu)$$

is of power $\aleph_\alpha$, since $\overline{\overline{F_{\mu_\varrho}}} = \aleph_\alpha$ and $\overline{\overline{\bigcup_{F_\mu \in \mathcal{G}_\varrho} (F_{\mu_\varrho} \cap F_\mu)}} \leqq r \cdot \bar{\varrho} < \aleph_\alpha$.

Thus there exists a ν such that $x_\nu \in F_{\mu_\varrho} - ((\mathcal{G}_\varrho) \cup S_\varrho))$.
Let ν_ϱ be the least ν of this kind. Thus x_{ν_ϱ}, F_{μ_ϱ} are defined for every $\varrho<\omega_\alpha$ and it follows by induction on ϱ that

(4.9.4) $$x_{\nu_\varrho} \in F_{\mu_\varrho},\ x_{\nu_\varrho} \notin (\mathcal{G}_\varrho) \ \text{ and } \ x_{\nu_\sigma} \neq x_{\nu_\varrho},\ \mu_\sigma \neq \mu_\varrho \ \text{ for every } \ \sigma<\varrho<\omega_\alpha.$$

Put $B = \{x_{\nu_\varrho}\}_{\varrho<\omega_\alpha}$. Now we prove

(4.9.5) $$B \cap F_\mu \neq 0 \quad \textit{for every} \quad \mu<\omega_\alpha.$$

For if not, then there exists a least μ^0 of this kind, and by (4.9.4) there is a $\mu_\varrho > \mu^0$ in contradiction to the definition of F_{μ_ϱ}.

(4.9.6) $$\overline{\overline{B \cap F_\mu}} < r^{+} \quad \textit{for every} \quad \mu<\omega_\alpha.$$

[6] If such a μ does not exist, then we stop with the construction and obviously one can prove in the same way that S_ϱ assures property $\mathbf{B}(r^{+})$ as we shall prove it later for B.

For if not, then there exists a $\mu^0 < \omega_\alpha$, a subset $B' \subset B$ and an $x_{r_{\varrho_0}} \in B$ such that $\overline{B'} = r$, $B' + \{x_{r_{\varrho_0}}\} \subseteq F_{\mu_0}$ and $\sigma < \varrho_0$ for every $x_{r_\sigma} \in B'$, and this obviously contradicts (4. 9.4), since then $F_{\mu_0} \in G_{\varrho_0}$.

(4. 9. 5) and (4. 9. 6) just mean that the family $\mathfrak{F}$ possesses property $\mathbf{B}(r^+)$.

Remark. As we have already mentioned in Problem 3a) — for a special case — we do not know whether $\mathbf{M}(\aleph_{\beta+1}, \aleph_{\beta+1}, \aleph_\gamma) \to \mathbf{B}(\aleph_{\gamma+1})$ is true or not if β and γ satisfy the assumption (v), i. e. if $cf(\beta) = cf(\gamma)$ and $\gamma < \beta$.

Now we need the following

(*) Lemma 3. *Let S be a set, $\overline{S} = \aleph_{\alpha+1}$, and suppose that $\aleph_\alpha$ is regular. Then there exists a system $\mathfrak{S}$ of subsets of S satisfying the following conditions:*

$p(\mathfrak{S}) = \aleph_\alpha$, $\mathfrak{S}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$ and for an arbitrary $S' \subset S$ $(\overline{S'} = \aleph_{\alpha+1})$ there exists an $X \in \mathfrak{S}$ such that $X \subseteq S'$.

Lemma 3 is a theorem of A. Hajnal.[7]

Now turning to the case B) we are going to prove that *if $\aleph_\alpha$ is singular and $cf(\alpha) < \beta \leqq \alpha$, then the trivial result $\mathbf{M}(\aleph_\beta, \aleph_\alpha, \aleph_\alpha) \to \mathbf{B}(\aleph_{\beta+1})$ (see (4. 1)) is best-possible, i. e.*

(*) (4. 10) $$\mathbf{M}(\aleph_\beta, \aleph_\alpha, \aleph_\alpha) \not\to \mathbf{B}(\aleph_\beta) \quad \textit{if} \quad cf(\alpha) < \beta \leqq \alpha.$$

We are going to prove this only for the case $\beta = \alpha$, the proof can be carried out similarly in the other cases too.[8]

Proof of the case $\beta = \alpha$. Let $\mathfrak{F}_1$ be a family satisfying the following conditions:

(4. 10. 1) $\overline{\overline{\mathfrak{F}_1}} = \aleph_{cf(\alpha)+1}$, $p(\mathfrak{F}_1) = \aleph_\alpha$ and $F \cap F' = 0$ for every $F, F' \in \mathfrak{F}_1$, $F \neq F'$.

$\aleph_{cf(\alpha)}$ being regular, we can apply Lemma 3 with $\mathfrak{F}_1$ instead of S and we obtain that there exists a system $\mathfrak{S}$ of subfamilies $\mathfrak{X}$ of $\mathfrak{F}_1$ satisfying the following conditions:

(4. 10. 2) $\overline{\overline{\mathfrak{S}}} = \aleph_{cf(\alpha)+1}$, $p(\mathfrak{S}) = \aleph_{cf(\alpha)}$, $\mathfrak{S}$ possesses property $\mathbf{C}(2, \aleph_{cf(\alpha)})$ and if $\mathfrak{F}'$ is a subfamily of $\mathfrak{F}_1$ such that $\overline{\overline{\mathfrak{F}'}} = \aleph_{cf(\alpha)+1}$, then there exists an $\mathfrak{X} \in \mathfrak{S}$ for which $\mathfrak{X} \subseteq \mathfrak{F}'$.

Let $\mathfrak{S} = \{\mathfrak{X}_\mu\}_{\mu < \omega_{cf(\alpha)+1}}$ and $\mathfrak{F}_1 = \{F_\tau\}_{\tau < \omega_{cf(\alpha)+1}}$ be well-orderings of type $\omega_{cf(\alpha)+1}$ of $\mathfrak{S}$ and $\mathfrak{F}_1$, respectively. Let further $\{\alpha_\nu\}_{\nu < \omega_{cf(\alpha)}}$ be a monotone increasing sequence of type $\omega_{cf(\alpha)}$ of ordinal numbers less than α cofinal with α.

[7] See [6], Theorem 9.

[8] We mention that if $cf(\alpha) < cf(\beta)$ (especially, if β is of the first kind), then the theorem is easy and can be proved without using (*). But for the cases $cf(\beta) = cf(\alpha)$ we have to use the same complicated proof as for the case $\beta = \alpha$. It is possible that a simpler proof can be constructed in this case too, but we were unsuccessful in doing this.

By (4.10.2) $\mathfrak{X}_\mu \subseteq \mathfrak{F}_1$ and $\overline{\overline{\mathfrak{X}_\mu}} = \aleph_{cf(\alpha)}$ for every $\mu < \omega_{cf(\alpha)+1}$. Let $\mathfrak{X}_\mu = \{F^\mu_\nu\}_{\nu<\omega_{cf(\alpha)}}$ be a well-ordering of type $\omega_{cf(\alpha)}$ of $\mathfrak{X}_\mu$.

The set F^μ_ν — being an element of $\mathfrak{F}_1$ — is of power $\aleph_\alpha$, and so it can be split into the sum of $\aleph_\alpha$ disjoint subsets of power $\aleph_{\alpha_\nu}$, that means: there exists a sequence $\{F^\mu_\nu(\gamma)\}_{\gamma<\omega_\alpha}$ of type ω_α of subsets of F^μ_ν satisfying the following conditions:

(4.10.3) $\overline{\overline{F^\mu_\nu(\gamma)}} = \aleph_{\alpha_\nu}$ for every $\gamma < \omega_\alpha$, $F^\mu_\nu(\gamma_1) \cap F^\mu_\nu(\gamma_2) = 0$ for every $\gamma_1, \gamma_2 < \omega_\alpha$, $\gamma_1 \neq \gamma_2$, and $F^\mu_\nu = \bigcup_{\gamma<\omega_\alpha} F^\mu_\nu(\gamma)$ where $\mu < \omega_{cf(\alpha)+1}$, $\nu < \omega_{cf(\alpha)}$ are arbitrary.

Now, corresponding to every $\mu < \omega_{cf(\alpha)+1}$ we define a family $\mathfrak{F}_{2,\mu}$ as follows. First put $F^\mu(\gamma) = \bigcup_{\nu<\omega_{cf(\alpha)}} F^\mu_\nu(\gamma)$, and then put $\mathfrak{F}_{2,\mu} = \{F^\mu(\gamma)\}_{\gamma<\omega_\alpha}$.

We have, for every $\mu < \omega_{cf(\alpha)+1}$,

(4.10.4) $\overline{\overline{\mathfrak{F}_{2,\mu}}} = \aleph_\alpha$, $F^\mu(\gamma_1) \cap F^\mu(\gamma_2) = 0$ for $\gamma_1, \gamma_2 < \omega_\alpha$, $\gamma_1 \neq \gamma_2$, $p(\mathfrak{F}_{2,\mu}) = \aleph_\alpha$ and $(\mathfrak{F}_{2,\mu}) = (\mathfrak{X}_\mu)$.

In fact, the second statement follows from (4.10.1) and (4.10.3), the first one is a corollary of it, while the third and the fourth ones are consequences of (4.10.3), since

$$\overline{\overline{F^\mu(\gamma)}} = \sum_{\nu<\omega_{cf(\alpha)}} \overline{\overline{F^\mu_\nu(\gamma)}} = \sum_{\nu<\omega_{cf(\alpha)}} \aleph_{\alpha_\nu} = \aleph_\alpha \text{ for every } \mu < \omega_{cf(\alpha)+1}, \gamma < \omega_\alpha.$$

Now we put

$$\mathfrak{F} = \mathfrak{F}_1 \cup \bigcup_{\mu<\omega_{cf(\alpha)+1}} \mathfrak{F}_{2,\mu}.$$

We have

$$p(\mathfrak{F}) = \aleph_\alpha, \tag{4.10.5}$$

since $p(\mathfrak{F}_1) = p(\mathfrak{F}_{2,\mu}) = \aleph_\alpha$ for $\mu < \omega_{cf(\alpha)+1}$ by (4.10.1) and (4.10.4).

Taking into consideration that $\aleph_\alpha$ is singular and therefore $cf(\alpha)+1 < \alpha$, we get from (4.10.1) and (4.10.4)

$$\overline{\overline{\mathfrak{F}}} = \overline{\overline{\mathfrak{F}_1}} + \sum_{\mu<\omega_{cf(\alpha)+1}} \overline{\overline{\mathfrak{F}_{2,\mu}}} = \aleph_{cf(\alpha)+1} + \aleph_{cf(\alpha)+1} \cdot \aleph_\alpha = \aleph_\alpha. \tag{4.10.6}$$

We are going to prove that

(4.10.7) $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_\alpha)$.

Let F, F' be two elements of $\mathfrak{F}$ such that $F \neq F'$.

To see that $\overline{\overline{F \cap F'}} < \aleph_\alpha$, we distinguish four cases: (i) $F, F' \in \mathfrak{F}_1$, (ii) $F \in \mathfrak{F}_1$, $F' \in \mathfrak{F}_{2,\mu}$, (iii) $F \in \mathfrak{F}_{2,\mu}$, $F' \in \mathfrak{F}_{2,\mu}$, (iiii) $F \in \mathfrak{F}_{2,\mu}$, $F' \in \mathfrak{F}_{2,\mu'}$ for some $\mu \neq \mu'$.

In the cases (i) and (iii) F and F' are disjoint by (4.10.1) and (4.10.4), respectively. If (ii) holds, then — by (4.10.3) — either $F \cap F' = 0$ if $F \notin \mathfrak{X}_\mu$,

or if $F \in \mathfrak{X}_\mu$, then $F = F^\mu_\nu$ for a $\nu < \omega_{cf(\alpha)}$ and $F' = F''(\gamma)$ for a $\gamma < \omega_\alpha$ and $\overline{\overline{F \cap F'}} = \overline{\overline{F^\mu_\nu(\gamma)}} = \aleph_{\alpha_\nu} < \aleph_\alpha$.

Suppose now that (iiii) holds. By (4. 10. 2) we have $\overline{\overline{\mathfrak{X}_\mu \cap \mathfrak{X}_{\mu'}}} < \aleph_{cf(\alpha)}$, since $\mu \neq \mu'$. It is obvious that $(\mathfrak{F}) = \bigcup_{\tau < \omega_{cf(\alpha)+1}} F_\tau$, and so $F \cap F' = \bigcup_{\tau < \omega_{cf(\alpha)+1}} (F_\tau \cap (F \cap F'))$ but either $F_\tau \cap F$ or $F_\tau \cap F'$ is empty if $F_\tau \notin \mathfrak{X}_\mu \cap \mathfrak{X}_{\mu'}$, hence

$$F \cap F' = \bigcup_{F_\tau \in \mathfrak{X}_\mu \cap \mathfrak{X}_{\mu'}} (F_\tau \cap (F \cap F')).$$

Taking into consideration that by (4. 10. 3) $\overline{\overline{F_\tau \cap F}} < \aleph_\alpha$, it results that $\overline{\overline{F \cap F'}} < \aleph_\alpha$ in this case too.

It remains to prove that

(4. 10. 8) $\mathfrak{F}$ does not possess property $\mathbf{B}(\aleph_\alpha)$.

Let B be a set such that $\overline{\overline{B \cap F}} < \aleph_\alpha$ for every $F \in \mathfrak{F}$. Then, especially, corresponding to every $\tau < \omega_{cf(\alpha)+1}$ there exists a subscript $\nu(\tau) < \omega_{cf(\alpha)}$ such that

$$\overline{\overline{B \cap F_\tau}} < \aleph_{\alpha_{\nu(\tau)}}.$$

It results that there exists a subfamily $\mathfrak{F}'$ of $\mathfrak{F}_1$ and an ordinal number $\nu_0 < \omega_{cf(\alpha)}$ such that $\overline{\overline{\mathfrak{F}'}} = \aleph_{cf(\alpha)+1}$ and $\overline{\overline{B \cap F_\tau}} < \aleph_{\alpha_{\nu_0}}$ for every $F_\tau \in \mathfrak{F}'$. But, by (4. 10. 2), then there exists a $\mu_0 < \omega_{cf(\alpha)+1}$ such that $\mathfrak{X}_{\mu_0} \subseteq \mathfrak{F}'$. Thus we have $\overline{\overline{F^{\mu_0}_\nu \cap B}} < \aleph_{\alpha_{\nu_0}}$ for every $\nu < \omega_{cf(\alpha)}$, and so $\overline{\overline{B \cap (\mathfrak{X}_{\mu_0})}} \leqq \aleph_{cf(\alpha)} \cdot \aleph_{\alpha_{\nu_0}} < \aleph_\alpha$. But by (4. 10.4) $\mathfrak{F}_{2,\mu_0}$ consists of $\aleph_\alpha$ disjoint subsets of $(\mathfrak{X}_{\mu_0})$, consequently there is an $F \in \mathfrak{F}_{2,\mu_0} \subseteq \mathfrak{F}$ such that

$$B \cap F = 0.$$

Thus by (4. 10. 5)—(4. 10.8) the case $\beta = \alpha$ of (4. 10) is proved.

From (4. 3) and (4. 10) we obtain the following

(*) Corollary. *Suppose p is infinite. Then $\mathbf{M}(p, p, p) \to \mathbf{B}(p)$ holds if and only if p is a regular cardinal number.*

This should be compared with Bernstein's theorem cited as Theorem 2 in Section 3.

Remark. After (3. 1) we have stated without proof that $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ is not monotone increasing in p. This can be seen e. g. by the following examples:

$\mathbf{M}(\aleph_1, \aleph_0, \aleph_2) \to \mathbf{B}(\aleph_1)$ holds by (3. 2) but
$\mathbf{M}(\aleph_1, \aleph_1, \aleph_2) \not\to \mathbf{B}(\aleph_1)$ by (4. 1); or
$\mathbf{M}(\aleph_2, \aleph_0, \aleph_1) \to \mathbf{B}(\aleph_1)$ by (3. 2) but
(*) $\mathbf{M}(\aleph_2, \aleph_1, \aleph_1) \not\to \mathbf{B}(\aleph_1)$ by Theorem 3 and (3. 2) and
(*) $\mathbf{M}(\aleph_2, \aleph_2, \aleph_1) \not\to \mathbf{B}(\aleph_1)$ by (4. 5).

However, every example which disproves the monotonicity in question is such that $s>p$. Under the condition $s \leqq p$ — and these are the only genuine cases — the monotonicity seems to hold. Suppose namely that $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ is true, $s \leqq p$, and for the sake of simplicity suppose further that $m, p, r, s \geqq \aleph_0$, and suppose (∗).

Distinguish three cases: (i) $p<r$, (ii) $p=r$, (iii) $p>r$. If (i) holds, then by (3. 1) and (4. 1) $m^+ \leqq s$, hence again by (3. 1) and (4. 1) $\mathbf{M}(m, p', r) \rightarrow \mathbf{B}(s)$ is true for every $p'>p$.

If (ii) holds, then $m \leqq p$ by Theorem 3 and by (3. 2), and so $p'>m, r$ for every $p'>p$, hence $\mathbf{M}(m, p', r) \rightarrow \mathbf{B}(s)$ is true by (4. 2).

If (iii) holds, then the implication is again trivial if $m \leqq p$, and if $m>p$, then by Theorem 6 which will be proved in Section 6 $\mathbf{M}(m, p', r) \rightarrow \mathbf{B}(p^+)$ is true for every $p'>p$.

By a slight modification of the proof of Theorem 6 one can obtain the following theorem:

(∗) *If $\mathfrak{F}$ is a family, $p(\mathfrak{F})=p'$, $\overline{\overline{\mathfrak{F}}}=m$ and $\mathfrak{F}$ possesses property* $\mathbf{C}(2, r)$, *then there exists a set B such that $\overline{\overline{B \cap F}}=p$ for every $F \in \mathfrak{F}$, provided that the above-mentioned inequalities hold for the cardinal numbers in question.*

Put $\mathfrak{F}'=\{B \cap F\}_{F \in \mathfrak{F}}$. It is obvious that $\mathfrak{F}$ possesses property $\mathbf{B}(s)$, provided the same holds for $\mathfrak{F}'$, but $p(\mathfrak{F}')=p$, $\overline{\overline{\mathfrak{F}'}} \leqq m$ and $\mathfrak{F}'$ possesses property $\mathbf{C}(2, r)$, hence $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ implies $\mathbf{M}(m, p', r) \rightarrow \mathbf{B}(s)$ in this case too.

It is possible that one can find a simpler proof for the monotonicity which does not use the hypothesis (∗), but we were unsuccessful in doing this.

5. Generalization of Miller's inductive construction. Let $(\mathfrak{F})$ be a family and S a set, $(\mathfrak{F}) \subseteq S$.

Def. (5. 1) Let $\mathfrak{F}'$ be a subfamily of $\mathfrak{F}$ and put $S'=(\mathfrak{F}')$. $\mathfrak{F}'$ is said to be closed in $\mathfrak{F}$ with respect to the cardinal number t (or briefly t-closed in $\mathfrak{F}$) if $F \in \mathfrak{F}$ and $\overline{\overline{F \cap S'}} \geqq t$ implies that $F \in \mathfrak{F}'$.

It is obvious that if $\mathfrak{F}'$ is an arbitrary subfamily of $\mathfrak{F}$, then (the intersection of any number of t-closed subfamilies being t-closed) for every t there exists a minimal t-closed subfamily of $\mathfrak{F}$ containing $\mathfrak{F}'$. However, we need concrete constructions for t-closed families containing $\mathfrak{F}'$.

Def. (5. 2) We define the t, ε closure of $\mathfrak{F}'$ in $\mathfrak{F}$: $\operatorname{Clos}(\mathfrak{F}', \mathfrak{F}, t, \varepsilon)$ for every ε. First we define a sequence $\{\mathfrak{G}_\nu\}_{\nu<\omega_\varepsilon}$ of type ω_ε of subfamilies of $\mathfrak{F}$ by induction on ν as follows:

Put $\mathfrak{G}_0=\mathfrak{F}'$ and $S_0=(\mathfrak{G}_0)$. Suppose that the families $\mathfrak{G}_\mu$ as well as the sets S_μ are already defined for every $\mu<\nu$ for a $\nu<\omega_\varepsilon$.

Put $S_\nu^* = \bigcup_{\mu < \nu} S_\mu$, $\mathcal{G}_\nu = \mathcal{G}(S_\nu^*, t, \mathfrak{F})$ (where $\mathcal{G}$ is the function defined in (4.6)), and $S_\nu = (\mathcal{G}_\nu)$.

Thus $\mathcal{G}_\nu$ is defined for every $\nu < \omega_\varepsilon$. Now we put

$$\mathrm{Clos}\,(\mathfrak{F}', \mathfrak{F}, t, \varepsilon) = \bigcup_{\nu < \omega_\varepsilon} \mathcal{G}_\nu.$$

As an immediate consequence of the definition we get that

$$(\mathrm{Clos}\,(\mathfrak{F}', \mathfrak{F}, t, \varepsilon)) = \bigcup_{\nu < \omega_\varepsilon} S_\nu \quad \text{and} \quad \mathfrak{F}' \subseteq \mathrm{Clos}\,(\mathfrak{F}', \mathfrak{F}, t, \varepsilon).$$

We have:

(5.3) Clos $(\mathfrak{F}', \mathfrak{F}, t, \varepsilon)$ *is t-closed for every* $t < \aleph_{cf(\varepsilon)}$.[9]

PROOF. Let F be an element of $\mathfrak{F}$ such that $\overline{\overline{F \cap (\mathrm{Clos}\,(\mathfrak{F}', \mathfrak{F}, t, \varepsilon))}} \geqq t$. Then $\overline{\overline{F \cap S_{\nu_0}^*}} \geqq t$ for a suitable $\nu_0 < \omega_\varepsilon$ and thus $F \in \mathcal{G}_{\nu_0} \subseteq \mathrm{Clos}\,(\mathfrak{F}', \mathfrak{F}, t, \varepsilon)$.

In what follows in this section let $\mathfrak{F}$ be a fixed family, $(\mathfrak{F}) = S$. Suppose that $p(\mathfrak{F}) = p$, $\overline{\overline{\mathfrak{F}}} = m$, $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$, where the cardinal numbers m, p, q, r, s and t satisfy the following inequalities:

(°) $\qquad m > p,\ p \geqq \aleph_0,\ 2 \leqq q \leqq p^+,\ r < p,\ r^+ \leqq s \leqq p,\ r \leqq t < p.$

Every statement proved in this section depends on the assumption (°). We are going to use the notations $p = \aleph_\alpha$, $m = \aleph_\beta$, $r = \aleph_\gamma$, $s = \aleph_\delta$ alternatively (provided r and s are infinite).

DEF. (5.4) Let $\varepsilon(t)$ denote the index of the least $\aleph$ greater than t. ($\varepsilon(t) = 0$ if t is finite and $\aleph_{\varepsilon(t)} = t^+$ if t is infinite.) This means that $\aleph_{\varepsilon(t)}$ is always regular. Put briefly Clos $(\mathfrak{F}', t)$ for Clos $(\mathfrak{F}', \mathfrak{F}, t, \varepsilon(t))$.

We need the following

(*) LEMMA 4. *Let $\mathfrak{F}'$ be a subfamily of $\mathfrak{F}$, $\overline{\overline{\mathfrak{F}'}} = m' \geqq p$. Then* $\overline{\overline{\mathrm{Clos}(\mathfrak{F}', t)}} = m'$, *provided one of the following conditions* (α) *and* ($\alpha\alpha$) *holds:*

(α) *$r = t$ and the following condition does not hold:*

(vv) *There exist ordinal numbers β' and γ such that $m' = \aleph_{\beta'}$, $r = \aleph_\gamma$ and $cf(\beta') = cf(\gamma)$.*

($\alpha\alpha$) $r < t$.

(Note that in case r is finite, the hypothesis (*) can be omitted.)

PROOF. Let $\mathcal{G}_\nu$ denote the families defined in (5.2) corresponding to the given $\mathfrak{F}'$, t and $\varepsilon(t)$. First we are going to prove by induction on ν that

(1) $\overline{\overline{\mathcal{G}_\nu}} = m'$ and $\overline{\overline{S_\nu}} \leqq m'$ for every $\nu < \omega_{\varepsilon(t)}$.

[9] It would be easy to see that (5.3) holds under more general conditions too, but we do not need this. E.g., it is true that for every t either Clos $(\mathfrak{F}', \mathfrak{F}, t, 0)$ or Clos $(\mathfrak{F}', \mathfrak{F}, t, 1)$ is t-closed.

This is true for $\nu=0$, since $\overline{\overline{\mathfrak{G}_0}}=\overline{\overline{\mathfrak{F}'}}=m'$ by the assumption and $\overline{\overline{S_0}}=\overline{\overline{(\mathfrak{G}_0)}}\leqq p\cdot m'=m'$. Suppose that the theorem is proved for every $\mu<\nu$ where $\nu<\omega_{\varepsilon(t)}$.

Then $\overline{\overline{S_\nu^*}}\leqq\sum_{\mu<\nu}m'=m'\overline{\overline{\nu}}$. But by (5.4) $\overline{\overline{\nu}}<\aleph_0\cdot t$ and therefore $\overline{\overline{S_\nu^*}}\leqq m'\cdot\aleph_0\cdot t=m'$, since $t<p\leqq m'$.

Now we obtain from (4.8.1), (4.8.2), (4.8.3) and (5.2) that $\overline{\overline{\mathfrak{G}_\nu}}\leqq q^-\cdot\overline{\overline{\mathfrak{G}_\nu|S_\nu^*}}$, $\mathfrak{G}_\nu|S_\nu^*$ possesses property $\mathbf{C}(q,r)$ and $|\mathfrak{G}_\nu|S_\nu^*|\geqq t$. On the other hand, we have $q\leqq(m')^+$, since $q\leqq p^+$ and $p\leqq m'$.

Hence by Lemmas 1 and 2 each of the conditions (α) and ($\alpha\alpha$) implies that $\overline{\overline{\mathfrak{G}_\nu|S_\nu^*}}\leqq m'$. Consequently, we have $\overline{\overline{\mathfrak{G}_\nu}}\leqq p\cdot m'=m'$, since $q^-\leqq p$ if $q\leqq p^+$. Thus $\overline{\overline{\mathfrak{G}_\nu}}=m'$, since $\mathfrak{G}_\nu$ contains $\mathfrak{G}_0$, and similarly as for the case $\nu=0$ we obtain that $\overline{\overline{S_\nu}}\leqq m'$, and (1) is proved.

Using that $t<p$, $p\geqq\aleph_0$ implies $\aleph_{\varepsilon(t)}\leqq p$, we get from (1)

$$m'\leqq\overline{\overline{\operatorname{Clos}(\mathfrak{F}',t)}}\leqq\sum_{\nu<\omega_{\varepsilon(t)}}m'\leqq m'\cdot p=m'$$

and Lemma 4 is proved.

Let now $\mathfrak{F}=\{F_\varrho\}_{\varrho<\omega_\beta}$ be a well-ordering of type ω_β of the family $\mathfrak{F}$.

Now we are going to define the sequences $\{\mathfrak{F}'_\sigma(t)\}_{\sigma<\varphi}$, $\{\mathfrak{F}_\sigma(t)\}_{\sigma<\varphi}$ of type φ of subfamilies of $\mathfrak{F}$ as well as the sequence $\{S_\sigma(t)\}_{\sigma<\varphi}$ of subsets of S for a $\varphi\leqq\omega_\beta$ by induction on σ as follows:

Def. (5.5) Put $\mathfrak{F}'_0(t)=\{F_\varrho\}_{\varrho<\omega_\alpha}$, $\mathfrak{F}_0(t)=\operatorname{Clos}(\mathfrak{F}'_0(t),t)$, $S_0(t)=(\mathfrak{F}_0(t))$. Suppose that the families $\mathfrak{F}'_{\sigma'}(t)$, $\mathfrak{F}_{\sigma'}(t)$ and the sets $S_{\sigma'}(t)$ are already defined for every $\sigma'<\sigma$. Put

$$\mathfrak{F}^*_\sigma(t)=\bigcup_{\sigma'<\sigma}\mathfrak{F}_{\sigma'}(t),\quad S^*_\sigma(t)=\bigcup_{\sigma'<\sigma}S_{\sigma'}(t).$$

If there exists an index $\varrho<\omega_\beta$ such that $F_\varrho\notin\mathfrak{F}^*_\sigma(t)$, then put $\varrho_\sigma=\varrho$ for the least ϱ of this kind, if not, then put $\sigma=\varphi$.

If ϱ_σ exists, then put

$$\mathfrak{F}'_\sigma(t)=\mathfrak{F}^*_\sigma(t)\cup\{\mathfrak{F}_{\varrho_\sigma}\},\quad\mathfrak{F}_\sigma(t)=\operatorname{Clos}(\mathfrak{F}'_\sigma(t),t),\quad S_\sigma(t)=(\mathfrak{F}_\sigma(t)).$$

Finally, if ϱ_σ is defined for every $\sigma<\omega_\beta$, then put $\varphi=\omega_\beta$.

(5.6) As an immediate consequence of the definition we obtain the following results:

(5.6.1) $$\mathfrak{F}=\bigcup_{\sigma<\varphi}\mathfrak{F}_\sigma(t),$$

(5.6.2) $\mathfrak{F}^*_{\sigma'}(t)\subset\mathfrak{F}'_{\sigma'}(t)\subseteq\mathfrak{F}_{\sigma'}(t)\subseteq\mathfrak{F}^*_\sigma(t)$ *for every* $\sigma'<\sigma<\varphi$,

(5.6.3) $S_{\sigma'}(t)\subseteq S^*_\sigma(t)\subseteq S_\sigma(t)$ *for every* $\sigma'<\sigma<\varphi$,

(5.6.4) $\mathfrak{F}_\sigma(t)$ *is t-closed in* $\mathfrak{F}$ *for every* $\sigma<\varphi$ *by* (5.3) *and* (5.4).

DEF. (5.7) Put $\mathfrak{H}_\sigma(t)=\mathfrak{F}_\sigma(t)-\mathfrak{F}^*_\sigma(t)$ for every $\sigma<\varphi$.

By (5.5) and (5.6.1) we have

(5.7.1) $$\mathfrak{F}=\bigcup_{\sigma<\varphi}\mathfrak{H}_\sigma(t),$$

and by (5.6.2)

(5.7.2) $$\mathfrak{H}_\sigma(t)\cap\mathfrak{H}_{\sigma'}(t)=0 \quad \text{for every} \quad \sigma'<\sigma<\varphi.$$

Now we prove the following lemma:

(5.8) *Suppose that $F_\varrho\in\mathfrak{H}_\sigma(t)$ for some $\varrho<\omega_\beta$, $\sigma<\varphi$. Then*

(β) $\overline{\overline{F_\varrho\cap S^*_\sigma(t)}}\leqq t$, *and if t is finite, then*

($\beta\beta$) $\overline{\overline{F_\varrho\cap S^*_\sigma(t)}}<t$.

PROOF. First of all — $\mathfrak{F}_{\sigma'}(t)$ being t-closed by (5.6.4) — we may suppose $\overline{\overline{F_\varrho\cap S_{\sigma'}(t)}}<t$ for every $\sigma'<\sigma$, for if not, then by the definition (5.1) F_ϱ belongs to $\mathfrak{F}_{\sigma'}(t)$ in contradiction to (5.7).

We distinguish two cases: (i) $\sigma=\sigma_1+1$ for a $\sigma_1<\sigma$, (ii) σ is of the second kind.

(i) By (5.5) and (5.6.3) we have $S^*_\sigma(t)=S_{\sigma_1}(t)$, hence ($\beta\beta$) holds for every t.

(ii) Let ω_τ be the least ordinal number cofinal with σ and let $\{\sigma_\eta\}_{\eta<\omega_\tau}$ be a monotone increasing sequence of ordinal numbers less than σ of type ω_τ cofinal with σ. We distinguish again two cases: (j) $\aleph_\tau\leqq t$, (jj) $\aleph_\tau>t$.

(j) We have by (5.5) and (5.6.3)

$$S^*_\sigma(t)=\bigcup_{\sigma'<\sigma}S_{\sigma'}(t)=\bigcup_{\eta<\omega_\tau}S_{\sigma_\eta}(t).$$

Hence $\overline{\overline{F_\varrho\cap S^*_\sigma(t)}}\leqq\sum_{\eta<\omega_\tau}\overline{\overline{S_{\sigma_\eta}\cap F_\varrho}}\leqq t\cdot\aleph_\tau=t$ and thus (β) holds.

(jj) Using again $S^*_\sigma(t)=\bigcup_{\eta<\omega_\tau}S_{\sigma_\eta}(t)$, we obtain that ($\beta\beta$) holds, for if not, then $F_\varrho\cap S^*_\sigma(t)$ contains a subset of power t which — $\aleph_\tau$ being regular — is contained already in a set $S_{\sigma_{\eta_0}}$ for an $\eta_0<\tau$.

If t is finite, then either (i) or (jj) holds for it, and therefore if t is finite, then ($\beta\beta$) is true.

DEF. (5.9) By (5.7.1) and (5.7.2) corresponding to every $\varrho<\omega_\beta$ there exists exactly one $\sigma<\varphi$ such that $F_\varrho\in\mathfrak{H}_\sigma(t)$. Put $\tilde{F}_\varrho=F_\varrho-S^*_\sigma(t)$ for this σ and put further $\tilde{\mathfrak{H}}_\sigma(t)=\{\tilde{F}_\varrho\}_{F_\varrho\in\mathfrak{H}_\sigma(t)}$. Put finally $\tilde{S}_\sigma(t)=(\tilde{\mathfrak{H}}_\sigma(t))$.

We need the following results:

It results from the assumption $p(\mathfrak{F})=p>t$ by (5.8) and (5.9) that

(5.10.1) *$p(\tilde{\mathfrak{H}}_\sigma(t))=p$ for every $\sigma<\varphi$*, and it is obvious from (5.9) that

(5.10.2) *the family $\tilde{\mathfrak{H}}_\sigma(t)$ possesses property $\mathbf{C}(q,r)$ for every $\sigma<\varphi$.*

(5. 10. 3) *Suppose* $F_\varrho \in \mathcal{H}_\sigma(t)$. *Then*

(γ) $\overline{\overline{F_\varrho \cap \bigcup_{\sigma' < \sigma} \tilde{S}_{\sigma'}(t)}} \leqq t$ *and the equality is excluded if* t *is finite, and*

($\gamma\gamma$) $F_\varrho \cap \tilde{S}_{\sigma''}(t) = 0$ *for every* $\sigma'' > \sigma$.

Proof. (γ) By the definitions (5.7) and (5.9) $\tilde{S}_{\sigma'}(t) \subset S_{\sigma'}(t) \subseteq S^*_\sigma(t)$ for every $\sigma' < \sigma$, hence by (5.8) we get $\overline{\overline{F_\varrho \cap \bigcup_{\sigma' < \sigma} \tilde{S}_{\sigma'}(t)}} \leqq \overline{\overline{F_\varrho \cap S^*_\sigma(t)}} \leqq t$ (or $< t$ if t is finite).

($\gamma\gamma$) It is enough to see that $F_\varrho \cap \tilde{F}_{\varrho''} = 0$ for every $\tilde{F}_{\varrho''} \in \tilde{\mathcal{H}}_{\sigma''}(t)$. But $F_\varrho \subseteq S_\sigma(t) \subseteq S^*_{\sigma''}(t)$, and so by (5.9) $\tilde{F}_{\varrho''} \cap F_\varrho \subseteq F'_{\varrho''} \cap S^*_{\sigma''} = 0$.

Now we prove the following

Lemma 5. *Suppose that the families* $\tilde{\mathcal{H}}_\sigma(t)$ *possess property* $\mathbf{B}(s)$ *for every* $\sigma < \varphi$. *Then the family* $\mathfrak{F}$ *possesses property* $\mathbf{B}(t^+ + s)$, *and if* t *is finite, then it possesses property* $\mathbf{B}((t-1)+s)$ *too.*

Proof. By the assumption for every $\sigma < \varphi$ there exists a set B_σ such that $B_\sigma \subseteq \tilde{S}_\sigma(t)$ and $1 \leqq \overline{\overline{B_\sigma \cap \tilde{F}_\varrho}} < s$ for every $\tilde{F}_\varrho \in \tilde{\mathcal{H}}_\sigma(t)$.

Put $B = \bigcup_{\sigma < \varphi} B_\sigma$. By (5.7.1) for every $\varrho < \omega_\beta$ there exists a $\sigma < \varphi$ such that $F_\varrho \in \mathcal{H}_\sigma(t)$. Then $\tilde{F}_\varrho \in \tilde{\mathcal{H}}_\sigma(t)$, $\tilde{F}_\varrho \subseteq F_\varrho$, by (5.9), and B_σ intersects $\tilde{F}_\varrho$ by the assumption, hence we get

$$B \cap F_\varrho \neq 0 \quad \text{for every} \quad \varrho < \omega_\beta. \tag{1}$$

Now we are going to prove that

$$\overline{\overline{B \cap F_\varrho}} < t^+ + s \quad \text{for every} \quad \varrho < \omega_\beta. \tag{2}$$

Let now σ_ϱ be the uniquely determined ordinal number for which $F_\varrho \in \mathcal{H}_{\sigma_\varrho}(t)$. By the definition of B we have

$$\overline{\overline{B \cap F_\varrho}} \leqq \overline{\overline{\bigcup_{\sigma < \sigma_\varrho} (B_\sigma \cap F_\varrho)}} + \overline{\overline{B_{\sigma_\varrho} \cap F_\varrho}} + \overline{\overline{\bigcup_{\sigma > \sigma_\varrho} (B_\sigma \cap F_\varrho)}}. \tag{x}$$

Taking into consideration that $B_\sigma \subseteq \tilde{S}_\sigma(t)$, we obtain from (5.10.3) that $\overline{\overline{\bigcup_{\sigma < \sigma_\varrho} (B_\sigma \cap F_\varrho)}} \leqq \overline{\overline{\bigcup_{\sigma < \sigma_\varrho} (\tilde{S}_\sigma(t) \cap F_\varrho)}} \leqq t$ and $\overline{\overline{\bigcup_{\sigma > \sigma_\varrho} (B_\sigma \cap F_\varrho)}} = 0$. On the other hand, it results from (5.9) that $\tilde{S}_{\sigma_\varrho}(t) \cap F_\varrho = \tilde{F}_\varrho$ for every $F_\varrho \in \mathcal{H}_{\sigma_\varrho}(t)$, hence $\overline{\overline{B \cap F_\varrho}} = \overline{\overline{B_{\sigma_\varrho} \cap \tilde{F}_\varrho}} < s$. It follows that $\overline{\overline{B \cap F_\varrho}} < t^+ + s$ for every $\varrho < \omega_\beta$. (1) and (2) mean that $\mathfrak{F}$ possesses property $\mathbf{B}(t^+ + s)$. Suppose now that t is finite. The formula (x) holds in this case too. We get from (5.10.3) that the first cardinal number on the right-hand side is less than t and the third one is 0, while the second is by the assumption less than s in this case too. Now if s is infinite, then the sum is less than s, hence less than $(t-1)+s$. If s is

finite, then the first summand being less than t is at most $t-1$, hence the sum is less than $(t-1)+s$ in this case too. It results from (1) that if t is finite, then $\mathfrak{F}$ possesses property $\mathbf{B}((t-1)+s)$.

LEMMA 6. *The family $\mathfrak{F}$ possesses property* $\mathbf{B}$, *provided the same holds for the families $\tilde{\mathfrak{H}}_\sigma(t)$ for every $\sigma<\varphi$.*

PROOF. Lemma 6 is to be seen quite similarly to Lemma 5. Let $B_\sigma \subseteq \tilde{S}_\sigma(t)$ denote the sets satisfying the condition $B_\sigma \cap \tilde{F}_\varrho \neq 0$, $\tilde{F}_\varrho \subsetneqq B_\sigma$ for every $\tilde{F}_\varrho \in \tilde{\mathfrak{H}}_\sigma(t)$. Put $B = \bigcup_{\sigma\ \varphi} B_\sigma$. The proof of the fact that B intersects every F_ϱ is the same as in Lemma 5. Let σ_ϱ denote, as before, the uniquely determined σ for which $\tilde{F}_\varrho \in \tilde{\mathfrak{H}}_{\sigma_\varrho}(t)$. It results from the definition (5.9) and from (5.10.3) that $B \cap \tilde{F}_\varrho = B_{\sigma_\varrho} \cap \tilde{F}_\varrho$, hence $\tilde{F}_\varrho \neq B \cap \tilde{F}_\varrho$, since $\tilde{F}_\varrho \subsetneqq B_{\sigma_\varrho}$, and thus $\tilde{F}_\varrho \subsetneqq B$, therefore $F_\varrho \subsetneqq B$ for every $\varrho<\varphi$.

For the sake of brevity we introduce the following notations:

DEF. (5.11) The cardinal number m is said to possess property $\mathbf{T}(p, r)$ if there exists an m' $(p \leq m' < m)$ such that m' satisfies the formula (vv) of Lemma 4, i. e. if there exist ordinal numbers β' and γ such that

$$m' = \aleph_{\beta'}, \quad r = \aleph_\gamma \quad \text{and} \quad cf(\beta') = cf(\gamma).$$

Quite similarly, p is said to possess property $\mathbf{Q}(r)$ if p satisfies the formula (v) of (4.9), i. e. if there exist ordinal numbers α_1 and γ such that

$$p = \aleph_\alpha = \aleph_{\alpha_1+1}, \quad r = \aleph_\gamma, \quad cf(\alpha_1) = cf(\gamma) \quad \text{and} \quad \gamma < \alpha_1.$$

Now we are going to prove

(∗) LEMMA 7. *$p(\overline{\tilde{\mathfrak{H}}_\sigma(t)}) = p$, the families $\tilde{\mathfrak{H}}_\sigma(t)$ possess property* $\mathbf{C}(q, r)$ *and $\overline{\tilde{\mathfrak{H}}_\sigma(t)} < m$ for every $\sigma < \varphi$, provided one of the conditions (δ) and $(\delta\delta)$ holds:*

(δ) $r = t$ and m does not possess property $\mathbf{T}(p, r)$.

$(\delta\delta)$ $r < t$.

(If t is finite, the hypothesis (∗) is not used.)

PROOF. The first two statements were proved in (5.10.1) and (5.10.2). We have to prove the third one. It is obvious from the definitions (5.7) and (5.9) that $\overline{\tilde{\mathfrak{H}}_\sigma(t)} \leq \overline{\mathfrak{H}_\sigma(t)} \leq \overline{\mathfrak{F}_\sigma(t)}$. We prove by induction on σ that $\overline{\mathfrak{F}_\sigma(t)} \leq$ $\leq p \cdot \overline{\sigma+1} < m$ for every $\sigma < \varphi$.

By the definition (5.5) $\overline{\overline{\mathfrak{F}_0'(t)}} = \aleph_\alpha = p$ and, since by the assumption either $r<t$ or m possesses property $\mathbf{T}(p, r)$, by Lemma 4 $\overline{\mathfrak{F}_0(t)} =$ $= \overline{\overline{\mathrm{Clos}\,(\mathfrak{F}_0'(t), t)}} = p$.

Suppose that we have $\overline{\overline{\mathfrak{F}_{\sigma'}(t)}} \leqq p \cdot \overline{\overline{\sigma'+1}}$ for every $\sigma' < \sigma$ for a $0 < \sigma < \varphi$. Then by (5.5)

$$\overline{\overline{\mathfrak{F}_\sigma^*}} \leqq \sum_{\sigma' < \sigma} \overline{\overline{\mathfrak{F}_{\sigma'}(t)}} \leqq \sum_{\sigma' < \sigma} p \cdot \overline{\overline{\sigma'+1}} = p \cdot \overline{\overline{\sigma}}.$$

Now $\overline{\overline{\mathfrak{F}_\sigma'(t)}} = \overline{\overline{\mathfrak{F}_\sigma^*(t)}} + 1 \leqq p \cdot \overline{\overline{\sigma+1}}$.

We have $\varphi \leqq \omega_\beta$ from the definition (5.5), and therefore $p \cdot \overline{\overline{\sigma+1}} < m$, hence we may apply Lemma 4 again to $\mathfrak{F}_\sigma(t) = \mathrm{Clos}\,(\mathfrak{F}_\sigma'(t), t)$ and we obtain $\overline{\overline{\mathfrak{F}_\sigma(t)}} \leqq p \cdot \overline{\overline{\sigma+1}}$, thus this statement is proved for every $\sigma < \varphi$ and Lemma 7 is proved.

Note that from the statement $\overline{\overline{\mathfrak{F}_\sigma(t)}} \leqq p \cdot \overline{\overline{\sigma+1}}$ $(\sigma < \varphi)$ it results that $\varphi = \omega_\beta$, but we do not use this fact.

Finally, to have a view of our results we need the following quite evident

LEMMA 8. *The least cardinal number which possesses property* $\mathbf{T}(\aleph_\alpha, \aleph_\gamma)$ $(\alpha > \gamma)$ *is* $\alpha + 1$ *if* $cf(\alpha) = cf(\gamma)$, *and it is* $\aleph_{\alpha + \omega_{cf(\gamma)}+1}$ *if* $cf(\alpha) \neq cf(\gamma)$.

PROOF. By the definition (5.11) we have to find the least β_1 for which there exists a β' such that $\alpha \leqq \beta' < \beta_1$ and $cf(\beta') = cf(\gamma)$. It is obvious that $\beta_1 = \beta' + 1$ for the least ordinal number β' satisfying this condition, and $\beta' = \alpha$ if $cf(\alpha) = cf(\gamma)$.

Suppose now $cf(\alpha) \neq cf(\gamma)$. $\beta' > \alpha$ has the form $\beta' = \alpha + \beta''$ and $cf(\alpha + \beta'') = cf(\gamma)$ can hold only if β'' is of the second kind. But then $cf(\alpha + \beta'') = cf(\beta'')$ and the least ordinal number β'' of the second kind satisfying $cf(\beta'') = cf(\gamma)$ is $\omega_{cf(\gamma)}$.

Let for the sake of brevity $\tau(\alpha, \gamma)$ denote the index of the least cardinal number which possesses property $\mathbf{T}(\aleph_\alpha, \aleph_\gamma)$.

EXAMPLES.

$$\tau(n, 0) = \omega + 1, \quad \tau(\omega, 0) = \omega + 1, \quad \tau(\omega + 1, 0) = \omega \cdot 2 + 1;$$

or more generally

$\tau(\alpha + \mu, \gamma) = \alpha + \omega_\gamma + 1$ for $1 \leqq \mu \leqq \omega_\gamma$ if $\gamma \leqq \alpha$ and ω_γ is regular.

6. Proof of the results concerning the conjectures (o) and (oo).

(*) THEOREM 4. *Suppose* $p \geqq \aleph_0$, $2 \leqq q \leqq p^+$ *and* $r^+ < p$. *Then for every cardinal number* m.

$$\mathbf{M}(m, p, q, r) \rightarrow \mathbf{B}.$$

(Note that if r is finite, the hypothesis (*) is not used.)

PROOF. For $m < p$ the theorem follows from Theorem 2 (BERNSTEIN'S theorem) if we use that symbol-I is decreasing in m (by (3.1)). We prove

it by induction on m for every $m>p$. Suppose that the theorem is true for every $m'<m$. Let now $\mathfrak{F}$ be a family $(p(\mathfrak{F})=p,\ \bar{\bar{\mathfrak{F}}}=m)$ which possesses property $\mathbf{C}(q,r)$.

Put $t=r^+$. Then the conditions $(^\circ)$ are satisfied for the cardinal numbers in question and $r<t$. Hence we can carry out the construction described in Section 5 and we can apply Lemma 7. It results that the families $\tilde{\mathfrak{H}}_\sigma(t)$ possess property $\mathbf{C}(q,r)$, $p(\tilde{\mathfrak{H}}_\sigma(t))=p$ and $\bar{\bar{\tilde{\mathfrak{H}}}}_\sigma(t)<m$ for every $\sigma<\varphi$. Using the induction hypothesis we obtain that the families $\tilde{\mathfrak{H}}_\sigma(t)$ possess property $\mathbf{B}$ and thus by Lemma 6 the same holds for the family $\mathfrak{F}$ too. Q. e. d.

REMARK. Theorem 4 is clearly a generalization of Theorem 1 (MILLER's theorem) for infinite r's, however, it is not best-possible in r as we have already mentioned. It is possible that under the conditions $p\geqq\aleph_0$, $q\leqq p^+$ the theorem holds for every $r<p$. We have to deal only with the case $p=r^+$.

Here we can prove the following

(*) THEOREM 5. *Suppose* $r=\aleph_\gamma$, $r^+=p$ *(i. e.* $p=\aleph_\alpha=\aleph_{\gamma+1}$*)*, $2\leqq q\leqq p^+$. *Then* $\mathbf{M}(m,p,q,r)\to\mathbf{B}$ *holds for every* m *less than* $\aleph_{\gamma+\omega_{cf(\gamma)}+1}$.

PROOF. For $m\leqq p$ the theorem is true by Theorem 2. We prove it by induction on m for every $p<m<\aleph_{\gamma+\omega_{cf(\gamma)}+1}$. Suppose that it is true for every $m'<m$ for an m satisfying the above condition. Let $\mathfrak{F}$ be a family for which $p(\mathfrak{F})=p$, $\bar{\bar{\mathfrak{F}}}=m$ and suppose that $\mathfrak{F}$ possesses property $\mathbf{C}(q,r)$. Put $t=r$. The conditions $(^\circ)$ hold for the cardinal numbers in question, and so we can consider the families $\tilde{\mathfrak{H}}_\sigma(r)$ $(\sigma<\varphi)$ defined in (5.9). Since by the assumption $cf(\alpha)=cf(\gamma+1)$ $(cf(\alpha)\neq cf(\gamma))$, it follows from Lemma 8 that m does not possess property $\mathbf{T}(p,r)$. It results from Lemma 7 $(\delta\delta)$ that $p(\tilde{\mathfrak{H}}_\sigma(r))=p$, $\tilde{\mathfrak{H}}_\sigma(r)$ possesses property $\mathbf{C}(q,r)$ and $\bar{\bar{\tilde{\mathfrak{H}}}}_\sigma(r)<m$ for every $\sigma<\varphi$. Hence by the induction hypothesis the families $\tilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{B}$. Consequently, by Lemma 6, the same is true for $\mathfrak{F}$.

REMARK. We do not know for any γ whether the assumption $m<\aleph_{\gamma+\omega_{cf(\gamma)}+1}$ can be omitted. We have formulated the simplest unsolved problem in Section 3 (see Problem 2).

(*) THEOREM 6. *Suppose* $p>r\geqq\aleph_0$, *then* $\mathbf{M}(m,p,r)\to\mathbf{B}(r^{++})$ *for every* m.

PROOF. If $p=r^+$, then the theorem is trivially true by (3.2). Thus we may suppose $r^+<p$. In the cases $m<p$ by (4.2) we have $\mathbf{M}(m,p,r)\to\mathbf{B}(2)$.

If p does not possess property $\mathbf{Q}(r)$, then by (4.9) $\mathbf{M}(p,p,r)\to\mathbf{B}(r^+)$ holds. If p possesses property $\mathbf{Q}(r)$, then it obviously does not possess roperty $\mathbf{Q}(r^+)$ (since if $r=\aleph_\gamma$, then $r^+=\aleph_{\gamma+1}$ and $cf(\gamma)\neq cf(\gamma+1)$).

It follows again from (4.9) that $\mathbf{M}(p, p, r^+) \rightarrow \mathbf{B}(r^{++})$ holds. As a consequence of (3. 1) we get that $\mathbf{M}(p, p, r) \rightarrow \mathbf{B}(r^{++})$ holds in every cases. Now we prove the theorem for $m > p$ by induction on m as follows:

Suppose that it is true for every $m' < m$. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F}) = p,\ \mathfrak{F} = m)$ which possesses property $\mathbf{C}(2, r)$. Put $t = r^+$. Then the conditions (°) hold for the cardinal numbers in question and we can consider the families $\tilde{\mathfrak{H}}_\sigma(t)$ $(\sigma < \varphi)$. Since $r < t$, it results from Lemma 7 that $p(\tilde{\mathfrak{H}}_\sigma(t)) = p$, the families $\tilde{\mathfrak{H}}_\sigma(t)$ possess property $\mathbf{C}(2, r)$ and $\tilde{\mathfrak{H}}_\sigma(t) < m$ for every $\sigma < \varphi$. Thus by the induction hypothesis the families $\tilde{\mathfrak{H}}_\sigma(t)$ possess property $\mathbf{B}(r^{++})$. Applying Lemma 5 we obtain that $\mathfrak{F}$ possesses property $\mathbf{B}(r^{++} + r^{++})$, i. e. it possesses property $\mathbf{B}(r^{++})$.

REMARK. It is obvious from (3. 1) that under the conditions of Theorem 6 $\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(s)$ holds for every $s \geqq r^{++}$ too. In the case $q = 2$ Theorem 4 is a corrollary of Theorem 6. Similarly as in the case of Theorem 4, it is possible that Theorem 6 holds with r^+ instead of r^{++}.

(*) THEOREM 7. *Suppose $p > r \geqq \aleph_0$. (Put $p = \aleph_\alpha$ $r = \aleph_\gamma$.) Suppose further that p does not possess property $\mathbf{Q}(r)$. Then*

$$\mathbf{M}(m, p, r) \rightarrow \mathbf{B}(r^+)$$

for every $m < \aleph_{\alpha + \omega_{cf(\gamma)} + 1}$, provided $cf(\alpha) \neq cf(\gamma)$.

PROOF. For $m < p$ the theorem is a corollary of (4. 2). In the case $m = p$ we get from (4. 9) that $\mathbf{M}(p, p, r) \rightarrow \mathbf{B}(r^+)$ holds, since the assumption of our theorem assures that p and r do not satisfy the formula (v) of (4. 9).

We are going to prove our theorem for $m > p$ by induction on m as follows: Suppose that the theorem is true for every $m' < m$, for an m satisfying the above condition. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F}) = p,\ \mathfrak{F} = m)$ which possesses property $\mathbf{C}(2, r)$. Put $t = r$. The conditions (°) are satisfied, and so we can consider the families $\tilde{\mathfrak{H}}_\sigma(r)$. The assumption $cf(\alpha) \neq cf(\gamma)$ assures by Lemma 8 that m does not possess property $\mathbf{T}(p, r)$. Thus from Lemma 7 we obtain that $p(\tilde{\mathfrak{H}}(r)) = p$, the families $\tilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{C}(2, r)$ and $\tilde{\mathfrak{H}}_\sigma(r) < m$ for every $\sigma < \varphi$.

Thus, by the induction hypothesis, the families $\tilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{B}(r^+)$ and, consequently, by Lemma 5, the family $\mathfrak{F}$ possesses $\mathbf{B}(r^+ + r^+)$ Since r is supposed to be infinite, this means that $\mathfrak{F}$ possesses property $\mathbf{B}(r^+)$ too.

REMARKS. If p possesses property $\mathbf{Q}(r)$, we do not know whether the theorem is true for $m = p$. (See the remark after (4. 9) and Problem 3a).)

If $cf(\alpha) = cf(\gamma)$, then by (4.9) the theorem is true for $m \leq p$, but we do not know whether it is true for $m = p^+$ or not. The simplest unsolved problem here is $\mathbf{M}(\aleph_{\omega+1}, \aleph_\omega, \aleph_0) \to \mathbf{B}(\aleph_{\omega+1})$.

Here the difficulty is essentially the same as in Problem 3b). It is obvious from the remark made after (4.9) that a positive solution of Problem 3b) would imply the positive solution of the problem just stated as well as a positive solution of Problem 3a).

7. The discussion of symbol-II in the cases $r < \aleph_0$ $(p \geqq \aleph_0)$. Note that in the case $r < \aleph_0$ $(p \geqq \aleph_0)$ symbol-I is completely discussed by MILLER's theorem. The positive theorems concerning symbol-II will be proved by MILLER's method quite similarly as the theorems of Section 6.

THEOREM 8. a) $\mathbf{M}(\aleph_{\alpha+n}, \aleph_\alpha, r) \to \mathbf{B}((r-1)(n+1)+2)$ *if* r *is finite and* α *is arbitrary.*

b) $\mathbf{M}(m, \aleph_\alpha, r) \to \mathbf{B}(\aleph_0)$ *for every* m *and* α, *provided* $r < \aleph_0$.[10]

PROOF. a) We are going to prove the theorem by induction on n. For $n = 0$ the theorem is proved in (4.9). Suppose that it is true for an n and let $\mathfrak{F}$ be a family such that $p(\mathfrak{F}) = \aleph_\alpha$, $\mathfrak{F} = \aleph_{\alpha+n+1}$ and suppose that it possesses property $\mathbf{C}(2, r)$. It is obvious that the conditions (°) hold for the cardinal numbers in question and we can apply the construction of Section 5 with $t = r$ to our family $\mathfrak{F}$.

By Lemma 7, $p(\widetilde{\mathfrak{H}}_\sigma(r)) = p$, the families $\widetilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{C}(2, r)$ and $\widetilde{\mathfrak{H}}_\sigma(r) < \aleph_{\alpha+n+1}$ for every $\sigma < \varphi$. This means that $\widetilde{\mathfrak{H}}_\sigma(r) \leq \aleph_{\alpha+n}$ for every $\sigma < \varphi$ and — using (3.1) — we get from the induction hypothesis that the families $\widetilde{\mathfrak{H}}_\sigma(r)$ possess property $\mathbf{B}((r-1)(n+1)+2)$ for every $\sigma < \varphi$.

It follows from Lemma 5 that the family $\mathfrak{F}$ possesses property $\mathbf{B}((r-1) + (r-1)(n+1)+2)$, i. e. it possesses property $\mathbf{B}((r-1)(n+2)+2)$.

b) The proof can be carried out by induction on m using Lemmas 5 and 7 quite similarly as in the previous cases, and so we omit the proof.

REMARK. The hypothesis (*) is not used in the proof, since it is not used in the proof of Lemma 7 for the case of finite r.

With a slight modification of our construction it would be easy to prove the following

THEOREM 9. *Let* $\mathfrak{F}$ *be a family,* $p(\mathfrak{F}) = \aleph_\alpha$, $\mathfrak{F} = \aleph_{\alpha+n}$, *and suppose that it possesses property* $\mathbf{C}(2, r)$ *for a finite* r *where* α *is arbitrary.*

Let there be given a function $l(F)$ *which correlates to every* $F \in \mathfrak{F}$ *an integer* $l(F)$.

[10] Note that n denotes always a non-negative integer and r is supposed to be greater than 0.

Then there exists a set B such that

$$\overline{\overline{B \cap F}} = \max\left(l(F), (r-1)(n+1)+1\right) \quad \textit{for every} \quad F \in \mathfrak{F}.$$

In particular, if $l(F) \equiv (r-1)(n+1)+1$, then the set B intersects every F in exactly $(r-1)(n+1)+1$ points.

We omit the proof.

Now we are going to prove that Theorem 8 is best-possible in s.

(*) THEOREM 10. a) $\mathbf{M}(\aleph_{\alpha+n}, \aleph_\alpha, r) \dashrightarrow \mathbf{B}((r-1)(n+1)+1)$ *if r is finite and α is arbitrary.*

b) $\mathbf{M}(m, \aleph_\alpha, r) \dashrightarrow \mathbf{B}(l)$ *if $r>1$ is finite, α is arbitrary, $m \geqq \aleph_{\alpha+\omega}$ and l is an integer.*

PROOF. a) We have to prove that there exists a family $\mathfrak{F}$ satisfying the following conditions:

(1) $p(\mathfrak{F}) = \aleph_\alpha$.

(2) $\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+n}$.

(3) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$.

(4) If for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}$, then there exists an $F_0 \in \mathfrak{F}$ such that $\overline{\overline{F_0 \cap B}} \geqq (r-1)(n+1)+1$.

We are going to prove instead of this the following more general statement: There exists a family $\mathfrak{F}$ satisfying the conditions (1), (2), (3) and the following condition:

(5) There exist subfamilies $\mathfrak{F}_1$, $\mathfrak{F}_2$ of $\mathfrak{F}$ such that $\mathfrak{F}_1 \cup \mathfrak{F}_2 = \mathfrak{F}$, $\mathfrak{F}_1 \cap \mathfrak{F}_2 = 0$ and if for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}_1$, then there exists an $F_0 \in \mathfrak{F}_2$ such that $\overline{\overline{F_0 \cap B}} \geqq (r-1)(n+1)+1$.

It is obvious that (5) implies (4).

Put $(\mathfrak{F}) = S$. Obviously (1) and (2) imply $\overline{\overline{S}} \leqq \aleph_{\alpha+n}$. Thus we have:

(6) If there exists a family $\mathfrak{F}$ satisfying the conditions (1), (2), (3) and (5), then for an arbitrary set S' ($\overline{\overline{S'}} = \aleph_{\alpha+n}$) there exists a family $\mathfrak{F}'$ such that $(\mathfrak{F}') \subseteq S'$ and $\mathfrak{F}'$ satisfies the conditions (1), (2), (3) and (5) too.

We prove the existence of such a family $\mathfrak{F}$ by induction on n. For $n = 0$ the theorem is proved in (4.5) (see the remark after (4.5)).[11] Suppose that for an n there exists a family $\mathfrak{F}$ satisfying the formulas (1), (2), (3) and (5). Let S be a set, $\overline{\overline{S}} = \aleph_{\alpha+n+1}$. Then $\overline{\overline{[S]^{\aleph_{\alpha+n}}}} = \aleph_{\alpha+n+1}$ by the hypothesis (*). Let $\{A_\varrho\}_{\varrho<\omega_{\alpha+n+1}} = [S]^{\aleph_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n+1}$ of the set $[S]^{\aleph_{\alpha+n}}$.

[11] In case of finite r the construction given in (4.5) can be simplified as follows: Suppose that $\overline{\overline{\mathfrak{F}_1}} = r$ instead of $\overline{\overline{\mathfrak{F}_1}} = r^+$ and take for $\mathfrak{S}$ the system of all subsets X of $(\mathfrak{F}_1)$ satisfying the condition $\overline{\overline{X \cap F}} = 1$ for every $F \in \mathfrak{F}_1$ instead of the system $\mathfrak{S}$ defined in (4.5.4).

We are going to define a sequence $\{\mathfrak{F}_\varrho\}_{\varrho<\omega_{\alpha+n+1}}$ of type $\omega_{\alpha+n+1}$ of families $(\mathfrak{F}_\varrho)\subseteq S$ by induction on ϱ as follows:

Suppose that the families $\mathfrak{F}_{\varrho'}$ are defined for a $\varrho<\omega_{\alpha+n+1}$ in such a way that $\overline{(\mathfrak{F}_{\varrho'})}\leqq\aleph_{\alpha+n}$ for every $\varrho'<\varrho$. Then $\overline{A_\varrho\cup\bigcup\limits_{\varrho'<\varrho}(\mathfrak{F}_{\varrho'})}\leqq\aleph_{\alpha+n}$, hence we can define a subset S_ϱ of S such that

$$S_\varrho\subseteq S-(A_\varrho\cup\bigcup_{\varrho'<\varrho}(\mathfrak{F}_{\varrho'})) \text{ and } \overline{S_\varrho}=\aleph_{\alpha+n}. \tag{7}$$

By the induction hypothesis and by (6) there exists a family $\mathfrak{F}^*_\varrho$ satisfying the formulas (1), (2), (3) and (5) such that

$$(\mathfrak{F}^*_\varrho)\subseteq S_\varrho; \tag{8}$$

let $\mathfrak{F}^{1,*}_\varrho$ and $\mathfrak{F}^{2,*}_\varrho$ denote the families satisfying (5) instead of $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively.

Since $\mathfrak{F}^*_\varrho$ satisfies (2), we have $\overline{\overline{\mathfrak{F}^{2,*}_\varrho}}\leqq\aleph_{\alpha+n}$ and we may suppose that the equality holds. Let $\mathfrak{F}^{2,*}_\varrho=\{F^{\varrho,2,*}_\nu\}_{\nu<\omega_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n}$ of the family $\mathfrak{F}^{2,*}_\varrho$.

Since $\overline{A_\varrho}=\aleph_{\alpha+n}$, it is obvious that there exists a system $\mathfrak{S}_\varrho$ of subsets of A_ϱ satisfying the following conditions:

(9) $\overline{\overline{\mathfrak{S}_\varrho}}=\aleph_{\alpha+n}$, $\overline{X}=r-1$ for every $X\in\mathfrak{S}$ and $X\cap Y=0$ for every $X, Y\in\mathfrak{S}$, $X\neq Y$.

Let $\mathfrak{S}_\varrho=\{X^\varrho_\nu\}_{\nu<\omega_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n}$ of the set $\mathfrak{S}_\varrho$.

We define the families $\mathfrak{F}_\varrho$, $\mathfrak{F}^1_\varrho$, $\mathfrak{F}^2_\varrho$ by the following formulas:

$$\mathfrak{F}^1_\varrho=\mathfrak{F}^{1,*}_\varrho,\quad \mathfrak{F}^2_\varrho=\{X^\varrho_\nu\cup F^{\varrho,2,*}_\nu\}_{\nu<\omega_{\alpha+n}} \tag{10}$$

and

$$\mathfrak{F}_\varrho=\mathfrak{F}^1_\varrho\cup\mathfrak{F}^2_\varrho.$$

It is obvious that $(\mathfrak{F}_\varrho)\subseteq S_\varrho+A_\varrho$, hence $\overline{\overline{(\mathfrak{F}_\varrho)}}\leqq\aleph_{\alpha+n}$, and so $\mathfrak{F}_\varrho$ is defined for every $\varrho<\omega_{\alpha+n+1}$ and the formulas (7)—(10) are satisfied for every $\varrho<\omega_{\alpha+\nu+1}$.

Put

$$\mathfrak{F}=\bigcup_{\varrho<\omega_{\alpha+n+1}}\mathfrak{F}_\varrho,\quad \mathfrak{F}^1=\bigcup_{\varrho<\omega_{\alpha+n+1}}\mathfrak{F}^1_\varrho,\quad \mathfrak{F}^2=\bigcup_{\varrho<\omega_{\alpha+n+1}}\mathfrak{F}^2_\varrho. \tag{11}$$

Now we have to verify that $\mathfrak{F}$ satisfies (1), (2), (3) and (5) for $n+1$ instead of n.

$p(\mathfrak{F}^*_\varrho)=\aleph_\alpha$, since $\mathfrak{F}^*_\varrho$ satisfies (1), and thus it follows immediately from the definitions (9), (10) and (11) that

$$p(\mathfrak{F})=\aleph_\alpha. \tag{12}$$

It results immediately from (7) and (8) that

$$(\mathfrak{F}^1_{\varrho'}) \cap (\mathfrak{F}^1_{\varrho}) = 0 \quad \text{if} \quad \varrho' < \varrho < \omega_{\alpha+n+1}. \tag{13}$$

Thus, since the families $\mathfrak{F}^1_\varrho$ are non-empty, we have

$$\mathfrak{F} = \aleph_{\alpha+n+1}. \tag{14}$$

Now we prove:

(15) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$.

Let F, F' be two distinct elements of $\mathfrak{F}$. Then $F \in \mathfrak{F}_\varrho$ and $F' \in \mathfrak{F}_{\varrho'}$ for suitable ϱ and ϱ', respectively. We distinguish two cases: (i) $\varrho = \varrho'$, (ii) $\varrho \neq \varrho'$.

(i) If $F \in \mathfrak{F}^1_\varrho$, $F' \in \mathfrak{F}^1_\varrho$, then $F \cup F' < r$, since by (10) $\mathfrak{F}^1_\varrho = \mathfrak{F}^{1,*}_\varrho$ and $\mathfrak{F}^*_\varrho$ satisfies (3).

If $F \in \mathfrak{F}^1_\varrho$, $F' \in \mathfrak{F}^2_\varrho$, then $F \subseteq S_\varrho$, $F' = X^\varrho_\nu \cup F^{\varrho,2,*}_\nu$ for a suitable $\nu < \omega_{\alpha+n}$, but by (9) $X^\varrho_\nu \subset A_\varrho$, hence by (7) and (8) $F \cap F' = F \cap F^{\varrho,2,*}_\nu$ and $F \cap F' < r$ follows again from the fact that $\mathfrak{F}^*_\varrho$ satisfies (3).

If $F \in \mathfrak{F}^2_\varrho$, $F' \in \mathfrak{F}^2_\varrho$, then $F = X^\varrho_\nu \cup F^{\varrho,2,*}_\nu$, $F' = X^\varrho_{\nu'} \cup F^{\varrho,2,*}_{\nu'}$ for suitable ν', ν $(\nu \neq \nu')$, respectively. Using again that A_ϱ and S_ϱ are disjoint, we get

$$F \cap F' = (X^\varrho_\nu \cap X^\varrho_{\nu'}) \cup (F^{\varrho,2,*}_\nu \cap F^{\varrho,2,*}_{\nu'}).$$

Thus, using that by (9) $X^\varrho_\nu \cap X^\varrho_{\nu'} = 0$, we get by the same argument as above that $F \cap F' < r$ in this case too.

(ii) We may suppose $\varrho' < \varrho$. If $F \in \mathfrak{F}^1_\varrho$, then by (7), (8) and (10) F and F' are disjoint. If $F \in \mathfrak{F}^2_\varrho$, then $F = X^\varrho_\nu \cup F^{\varrho,2,*}_\nu$ for a suitable ν and it results from (7) and (8) that $F' \cap F \subset X^\varrho_\nu$, hence by (9) $F' \cap F \leq r-1 < r$.

We have

$$\mathfrak{F}^1 \cap \mathfrak{F}^2 = 0. \tag{16}$$

In fact, $\mathfrak{F}^{1,*}_\varrho \cap \mathfrak{F}^{2,*}_\varrho = 0$ for every ϱ, because $\mathfrak{F}^*_\varrho$ satisfies (5), thus it results from the definition (10) and e. g. from the fact that $\mathfrak{F}^*_\varrho$ satisfies (3), that $\mathfrak{F}^1_\varrho \cap \mathfrak{F}^2_\varrho = 0$ and it is obvious from (7) and (8) that $\mathfrak{F}_{\varrho'} \cap \mathfrak{F}_\varrho = 0$ for $\varrho' \neq \varrho$, hence $\mathfrak{F}_1 \cap \mathfrak{F}_2 = 0$ is true.

Now we prove:

(17) Suppose that for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}^1$.

Then there exists an $F_0 \in \mathfrak{F}_2$ such that $F_0 \cap B \geqq (r-1)(n+2)+1$.

First of all it follows from (13) that $B = \aleph_{\alpha+n+1}$. As a corollary of this there exists a subscript ϱ_0 such that $A_{\varrho_0} \subseteq B$. Since by the assumption B has to intersect every $F \in \mathfrak{F}^1$, we have that $F \cap B \neq 0$ for every $F \in \mathfrak{F}^1_{\varrho_0}$. But by (10) $\mathfrak{F}^1_\varrho = \mathfrak{F}^{1,*}_\varrho$ and it follows from the fact that $\mathfrak{F}^*_{\varrho_0}$, $\mathfrak{F}^{1,*}_{\varrho_0}$, $\mathfrak{F}^{2,*}_{\varrho_0}$ satisfy (5), that there exists an index ν_0 such that $B \cap F^{\varrho_0,2,*}_{\nu_0} \geqq (r-1)(n+1)+1$. Put

$F^0 = X^{\varrho_0}_{\nu_0} \cup F^{\varrho_0,2,*}_{\nu_0}$. Then $F^0 \in \mathfrak{F}^2$. Taking into consideration that by (7) and (8) $X^{\varrho_0}_{\nu_0} \cap F^{\varrho_0,2,*}_{\nu_0} = 0$ and by (9) $X^{\varrho_0}_{\nu_0} \subseteq A_{\varrho_0} \subseteq B$, $\overline{\overline{X^{\varrho}_{\nu_0}}} = r-1$, we obtain

$$\overline{\overline{B \cap F^0}} \geqq \overline{\overline{X^{\varrho_0}_{\nu_0}}} + \overline{\overline{B \cap F^{\varrho_0,2,*}_{\nu_0}}} \geqq (r-1)(n+2)+1.$$

Thus the families $\mathfrak{F}$, $\mathfrak{F}^1$ and $\mathfrak{F}^2$ satisfy by (12), (14), (15), (16) and (17) the formulas (1), (2), (3) and (5) for $n+1$ instead of n, and so the existence of such a family is proved for every n.

b) By (3.1) it suffices to prove that $\mathbf{M}(\aleph_{\alpha+\omega}, \aleph_\alpha, r) \dashrightarrow \mathbf{B}(l)$.

Let $\{S_n\}_{n<\omega}$ be a sequence of disjoint sets such that $\overline{\overline{S_n}} = \aleph_{\alpha+n}$. By the theorem just proved and by the remark (6) there exists a sequence $\{\mathfrak{F}_n\}_{n<\omega}$ of families such that $(\mathfrak{F}_n) \subseteq S_n$ and $\mathfrak{F}_n$ satisfies for every n the conditions (1), (2), (3) and (5).

Put $\mathfrak{F} = \bigcup_{n<\omega} \mathfrak{F}_n$. Then $p(\mathfrak{F}) = \aleph_\alpha$ and $\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+\omega}$, since the $\mathfrak{F}_n$'s satisfy (1) and (2) for every n and the $\mathfrak{F}_n$'s are obviously disjoint.

Since the sets S_n are disjoint, $F \cap F' = 0$, provided $F \in \mathfrak{F}_n$, $F' \in \mathfrak{F}_{n'}$ for $n \neq n'$. Thus, taking into consideration that $\mathfrak{F}_n$ satisfies (3) for every n, it follows that $\mathfrak{F}$ possesses property $\mathbf{C}(2, r)$. But $\mathfrak{F}$ does not possess property $\mathbf{B}(l)$ for any l, since there exists an n_0 such that $(r-1)(n_0+1)+1 > l$ and the subfamily $\mathfrak{F}_{n_0}$ of $\mathfrak{F}$ does not possess property $\mathbf{B}((n-1)(n_0+1)+1)$, because it satisfies (5).

Thus part b) of Theorem 10 is also proved.

REMARK. As we have already mentioned in (4.5), in the case $n=0$ of the part a) of Theorem 10 the hypothesis (*) is not used. We do not even know whether one can prove Theorem 10a) for $n=1$ without using (*).

8. Results on the topological products. A topological space $\mathfrak{X}$ is said to be $\varkappa$-compact if every family $\mathfrak{M}$ of closed subsets of it with void intersection, $\bigcap_{X \in \mathfrak{M}} X = 0$, contains a subfamily $\mathfrak{M}' \subseteq \mathfrak{M}$ $(\overline{\overline{\mathfrak{M}'}} < \aleph_\varkappa)$ with void intersection.

0-compactness means ordinary compactness.

1-compact spaces are the Lindelöf spaces.

For the sake of brevity we introduce the symbol $\boldsymbol{T}(m, \lambda) \to \varkappa$ to indicate the following statement:

If $\mathfrak{F}$ is a family of λ-compact *discrete* topological spaces, $\overline{\overline{\mathfrak{F}}} = m$, then the topological product of the elements of $\mathfrak{F}$ is $\varkappa$-compact.

As usual, $\boldsymbol{T}(m, \lambda) \not\to \varkappa$ denotes the negation of this statement.

TYCHONOV's classical theorem can be stated as follows: $\boldsymbol{T}(m, 0) \to 0$ for every cardinal number m.

Let S be a set, $\overline{\overline{S}}=m$, and let $\mu(x)$ be a measure defined on all subsets of S such that the values of $\mu(x)$ are 0 and 1, $\mu(\{x\})=0$ for every $x \in S$.

The cardinal number m is said to be of measure 0 if every σ-measure satisfying the above condition vanishes identically.[12]

A well-known result of ULAM states that every cardinal number m less than the first strongly inaccessible aleph is of measure 0.[13]

The hypothesis (**) states that a strongly inaccessible $>\aleph_0$ aleph is not of measure 0 or more generally:

(**) If m is strongly inaccessible, $>\aleph_0$, then there exists an m-additive measure satisfying the above conditions such that $\mu(S)=1$.

If we use (*), then ŁOS's theorem (Theorem 4 of [3]) states that

$$T(\aleph_{\varkappa+1}, 1) \not\to \varkappa \quad \text{for every} \quad \varkappa \geqq 1,$$

provided $\aleph_\varkappa$ is regular and of measure 0.[14]

Now we are going to prove the following

(*) THEOREM 11. *$T(\aleph_{\alpha+n}, \alpha+1) \not\to \alpha+n$ for every ordinal number α and for every $1 \leqq n < \omega$.*

Before proving this theorem[15] we compare it with ŁOS's theorem and state the simplest unsolved problems. Put $\alpha=0$, then our theorem states that $T(\aleph_n, 1) \not\to n$ for every $n \geqq 1$, and so it is stronger than ŁOS's theorem for the cases $\varkappa<\omega$. Moreover it is best-possible, namely $T(\aleph_n, 1) \to n+1$ is trivially true, since the topological product of $\aleph_\alpha$ Lindelöf spaces contains a base of power $\aleph_\alpha$ for every α. For the case of singular $\varkappa$'s, e. g. for $\varkappa=\omega$ the following problem remains open:

PROBLEM 4. $T(\aleph_\omega, 1) \to \omega$?

($T(\aleph_{\omega}, 1) \to \omega+1$ is trivially true and $T(\aleph_\omega, 1) \not\to n$ for every finite n is a consequence of both theorems.)

For $\varkappa$'s greater than ω ŁOS's theorem is stronger, since our result states nothing about $\varkappa$-compactness of the product of Lindelöf spaces for $\varkappa>\omega$.

But we do not know whether ŁOS's theorem is best-possible e. g. for $\varkappa=\omega+1$, since it states $T(\aleph_{\omega+2}, 1) \to \omega+1$ and the following problem remains open:

PROBLEM 5. $T(\aleph_{\omega+2}, 1) \to \omega+2$?

(Our Theorem 11 gives only that $T(\aleph_{\omega+2}, \omega+1) \not\to \omega+2$.)

[12] See [3], p. 14.

[13] See [7].

[14] See [3], Theorem 4, p. 17.

[15] The proof is given on p. 115.

For $\aleph_\varkappa$'s not less than the first inaccessible cardinal number Łos's theorem does not state anything. The reason for this is that if, at least, we assume the hypothesis (**), then $T(m_0, 1) \to \alpha_0$ is true where $m_0 = \aleph_{\alpha_0}$ denotes the first strongly inaccessible cardinal number $> \aleph_0$. More generally we have the following

(**) THEOREM 12. *If* $\aleph_\alpha$ *is strongly inaccessible,* $> \aleph_0$, *then*

$$T(\aleph_\alpha, \alpha) \to \alpha.$$ [16]

We mention here that even using (*) and (**) we can not decide whether $T(\aleph_\alpha, 1) \to \alpha_0$ is true if $\alpha > \alpha_0$ where $\aleph_{\alpha_0}$ is the first inaccessible cardinal number $> \aleph_0$.

Our theorem shows that $T(\aleph_\alpha + n, \alpha_0 + 1) \not\to \alpha_0 + n$ for every $1 \leq n < \omega$, but neither Łos's theorem nor our theorem disproves that $T(m, \alpha_0 + 1) \to \alpha_0 + \omega$ holds for every cardinal number m if $\aleph_{\alpha_0}$ is strongly inaccessible $> \aleph_0$.

PROOF OF THEOREM 11. Let r_0 be an integer such that $(r_0 - 1)(n+1) + 1 \geqq (r_0 - 1)n + 2$ (e. g. $r_0 = 2$). By Theorem 10 corresponding to every n there exists a family $\mathfrak{F}$ $((\mathfrak{F}) = S)$ satisfying the following conditions:

(1) $p(\mathfrak{F}) = \aleph_\alpha$.

(2) $\overline{\overline{\mathfrak{F}}} = \aleph_{\alpha+n}$.

(3) $\mathfrak{F}$ possesses property $\mathbf{C}(2, r_0)$.

(4) If for a set B $B \cap F \neq 0$ for every $F \in \mathfrak{F}$, then there exists an $F_0 \in \mathfrak{F}$, such that

$$\overline{\overline{F_0 \cap B}} \geqq (r_0 - 1)(n+1) + 1.$$

Let $\mathfrak{F} = \{F_\varrho\}_{\varrho < \omega_{\alpha+n}}$ be a well-ordering of type $\omega_{\alpha+n}$ of $\mathfrak{F}$. Let $\mathfrak{X}$ denote the topological product of the discrete spaces F_ϱ. The elements of $\mathfrak{X}$ are the sequences $(x_\varrho)_{\varrho < \omega_{\alpha+n}}$ where $x_\varrho \in F_\varrho$.

Corresponding to every finite sequence $\varrho_1 < \cdots < \varrho_k < \omega_{\alpha+n}$ we define the subset $B_{\varrho_1 \ldots \varrho_k}((x_\varrho)_{\varrho < \omega_{\alpha+n}})$ of S as the set of ϱ_ith components of $(x_\varrho)_{\varrho < \omega_{\alpha+n}}$ for $i = 1, \ldots, k$, i. e. we put

$$B_{\varrho_1 \ldots \varrho_k}((x_\varrho)_{\varrho < \omega_{\alpha+n}}) = \{x : x \in S \wedge (x = x_{\varrho_1} \vee \cdots \vee x = x_{\varrho_k})\}. \tag{5}$$

Now we define the subset $X_{\varrho_1 \ldots \varrho_k}$ of $\mathfrak{X}$ as follows:

$$(x_\varrho)_{\varrho < \omega_{\alpha+n}} \in X_{\varrho_1 \ldots \varrho_k} \quad \text{if and only if} \tag{6}$$

$$\overline{\overline{B_{\varrho_1 \ldots \varrho_k}((x_\varrho)_{\varrho < \omega_{\alpha+n}}) \cap F_{\varrho_i}}} < (r_0 - 1)n + 2 \quad \text{for every} \quad i = 1, \ldots, k.$$

[16] For the proof see p. 116.

Put

$$\mathfrak{M} = \{X_{\varrho_1 \dots \varrho_k}\}_{(\varrho_1 < \dots < \varrho_k < \omega_{\alpha+n})}.$$

It is obvious that $X_{\varrho_1 \dots \varrho_k}$ is a closed subset of $\mathfrak{X}$ for every sequence $\varrho_1 < \dots < \varrho_k < \omega_{\alpha+n}$ and it results from (1) that the discrete spaces F_ϱ are $\aleph_{\alpha+1}$-compact for every $\varrho < \omega_{\alpha+n}$. Hence it is enough to prove the following assertions:

(7) $$\bigcap_{X_{\varrho_1 \dots \varrho_k} \in \mathfrak{M}} X_{\varrho_1 \dots \varrho_k} = 0$$

and

(8) $$\bigcap_{X_{\varrho_1 \dots \varrho_k} \in \mathfrak{M}'} X_{\varrho_1 \dots \varrho_k} \neq 0 \quad \text{if} \quad \mathfrak{M}' \subseteq \mathfrak{M}, \quad \overline{\overline{\mathfrak{M}'}} < \aleph_{\alpha+n}.$$

Proof of (7). Let $(x^0_\varrho)_{\varrho < \omega_{\alpha+n}}$ be an arbitrary fixed element of $\mathfrak{X}$. Let B^0 be the set of those $x \in S$ for which there exists a $\varrho < \omega_{\alpha+n}$ such that $x = x^0_\varrho$. It is obvious that $B^0 \cap F_\varrho \neq 0$ for every $\varrho < \omega_{\alpha+n}$, hence by (4) we have for a $\varrho_0 < \omega_{\alpha+n}$

$$\overline{\overline{B^0 \cap F_{\varrho_0}}} \geqq (r_0 - 1)(n+1) + 1.$$

Put $(r_0 - 1)(n+1) + 1 = k_0$. Then there exists a sequence $\varrho^0_1 < \dots < \varrho^0_{k_0}$ such that $\varrho_0 = \varrho^0_{i_0}$ for an i_0 $(1 \leqq i_0 \leqq k_0)$, $\overline{\overline{\{x^0_{\varrho^0_i}\}}}_{1 \leqq i \leqq k_0} = k_0$ and $\{x^0_{\varrho^0_i}\}_{1 \leqq i \leqq k_0} \subseteq F_{\varrho_0} = F_{\varrho^0_{i_0}}$.

But this means that $\overline{\overline{B_{\varrho^0_1 \dots \varrho^0_{k_0}}((x^0_\varrho)_{\varrho < \omega_{\alpha+n}}) \cap F_{\varrho^0_{i_0}}}} = k_0 \lessdot (r_0 - 1)n + 2$ and thus by (6) $(x^0_\varrho)_{\varrho < \omega_{\alpha+n}} \notin X_{\varrho^0_1 \dots \varrho^0_{k_0}}$ which proves that the product considered in (7) is empty.

Proof of (8). Let $I(\mathfrak{M}')$ denote the set of ordinal numbers ϱ appearing as a subscript ϱ_i $(i = 1, \dots, k)$ of an $X_{\varrho_1 \dots \varrho_k} \in \mathfrak{M}'$. It is obvious that $X_{\varrho_1 \dots \varrho_k} \neq X_{\varrho'_1 \dots \varrho'_k}$ if the sequences $\varrho_1, \dots, \varrho_k$ and $\varrho'_1, \dots, \varrho'_k$ are different. Hence $\mathfrak{M}' \subset \mathfrak{M}$, $\overline{\overline{\mathfrak{M}'}} < \aleph_{\alpha+n}$ implies $\overline{\overline{I(\mathfrak{M}')}} < \aleph_{\alpha+n}$. Thus it is sufficient to see that

$$\bigcap_{\varrho_i (i=1,\dots,k),\, \varrho_i < \varrho_0} X_{\varrho_1 \dots \varrho_k} \neq 0 \quad \text{holds for every} \quad \varrho_0 < \omega_{\alpha+n}.$$

Put $\mathfrak{F}_{\varrho_0} = \{F_{\varrho'}\}_{\varrho' < \varrho_0}$ for every $\varrho_0 < \omega_{\alpha+n}$. Then $p(\mathfrak{F}_{\varrho_0}) = \aleph_\alpha$ by (1). $\mathfrak{F}_{\varrho_0}$ possesses property $\mathbf{C}(2, r_0)$ by (3) and $\overline{\overline{\mathfrak{F}_{\varrho_0}}} \leqq \aleph_{\alpha+n-1}$ $(n - 1 \geqq 0)$ for every $\varrho_0 < \omega_{\alpha+n}$. Thus by Theorem 8a) there exists a set B such that

$$1 \leqq \overline{\overline{B \cap F_{\varrho'}}} < (r_0 - 1)n + 2 \quad \text{for every} \quad \varrho' < \varrho_0.$$

It results that we can point out an element $x^0_{\varrho'}$ of $B \cap F_{\varrho'}$ for every $\varrho' < \varrho_0$ and let $x^0_{\varrho'}$ be an arbitrary element of $F_{\varrho'}$ for $\varrho' \geqq \varrho_0$. It is obvious from (6) that the sequence $(x^0_\varrho)_{\varrho < \omega_{\alpha+n}}$ so defined is an element of the product in question.

PROOF OF THEOREM 12. Let $\mathfrak{F}$ be a family, $\overline{\overline{\mathfrak{F}}} = \aleph_\alpha$ such that $\overline{\overline{F}} < \aleph_\alpha$ for every $F \in \mathfrak{F}$. Let $\mathfrak{F} = \{F_\nu\}_{\nu < \omega_\alpha}$ be a well-ordering of type ω_α of $\mathfrak{F}$. Put

$\mathfrak{F}_\nu = \{F_\mu\}_{\mu<\nu}$. Let $\mathfrak{X}$ and $\mathfrak{X}_\nu$ denote the topological product of the elements of $\mathfrak{F}$ and $\mathfrak{F}_\nu$, respectively. If $\Theta = (x_\nu^0)_{\nu<\omega_\alpha}$ is an element of $\mathfrak{X}$, then Θ/ν denotes the element $(x_\mu^0)_{\mu<\nu}$ of $\mathfrak{X}_\nu$.

Let there be given a family $\mathfrak{M}$ of closed subsets of $\mathfrak{X}$. Corresponding to every $X \in \mathfrak{M}$ and $\nu<\omega_\alpha$ we define a subset $Y(X, \nu)$ of $\mathfrak{X}_\nu$ as follows:

$$Y(X, \nu) = \{\Theta/\nu\}_{\Theta \in X}.$$

The set $\{Y(X, \nu)\}_{X \in \mathfrak{M}}$ is of power less than $\aleph_\alpha$ for every $\nu<\omega_\alpha$, since $\aleph_\alpha$ is strongly inaccessible and $\mathfrak{X}_\nu < \aleph_\alpha$ for every $\nu<\omega_\alpha$. As an easy consequence of this we obtain that $\bigcap_{X\in\mathfrak{M}} Y(X,\nu) \neq 0$ for every $\nu<\omega_\alpha$, provided $\bigcap_{X\in\mathfrak{M}'} X \neq 0$ for every $\mathfrak{M}' \subseteq \mathfrak{M}$, $\mathfrak{M}' < \aleph_\alpha$.

Put $Z_\nu = \bigcap_{X\in\mathfrak{M}} Y(X,\nu)$. The Z_ν's form a ramification system. By a result of P. Erdős and A. Tarski[17] it follows from the hypothesis (**) that there exists a $\Theta \in \mathfrak{X}$ such that $\Theta/\nu \in Z_\nu$ for every $\nu<\omega_\alpha$.

Let X be an arbitrary element of $\mathfrak{M}$. Then for an arbitrary $\nu<\omega_\alpha$ there exists a $\Theta_\nu \in X$ such that $\Theta_\nu/\nu = \Theta/\nu$. Since X is closed, it follows that $\Theta \in X$, and so $\Theta \in \bigcap_{X\in\mathfrak{M}} X$, i. e. $\mathfrak{X}$ is α-compact.

Now we state some unsolved problems which all would have been consequences of $T(\aleph_2, 1) \to 2$. The answer to all these questions is very likely negative, but we can not disprove any of them. In the formulation of all these problems we consider (*) to be assumed.

Problem 6. Let $\mathfrak{F}$ be a family ($\mathfrak{F} = \aleph_2$, $p(\mathfrak{F}) = \aleph_0$) such that every $\mathfrak{F}' \subseteq \mathfrak{F}$ ($\mathfrak{F}' \leqq \aleph_1$) possesses property **B**. Does then $\mathfrak{F}$ necessarily possess property **B** too?[18]

The family $\mathfrak{F}$ is said to possess property **G** if there exists a function $f(F)$ defined for every $F \in \mathfrak{F}$ such that $f(F)$ is an element of F and $f(F_1) \neq f(F_2)$ for $F_1 \neq F_2$.

Problem 7. Let $\mathfrak{F}$ be a family ($\mathfrak{F} = \aleph_2$, $p(\mathfrak{F}) = \aleph_0$) such that every $\mathfrak{F}' \subseteq \mathfrak{F}$ ($\mathfrak{F}' \leqq \aleph_1$) possesses property **G**. Does then $\mathfrak{F}$ necessarily possess property **G** too?[19]

[17] See the footnote [4] on p. 328 of [8].

[18] The following theorem is an easy consequence of Tychonov's theorem: If $\mathfrak{F}$ is a family of finite sets such that every finite subfamily of $\mathfrak{F}$ possesses property **B**, then $\mathfrak{F}$ possesses property **B**.

[19] This problem is due to W. Gustin (oral communication). It is well known and an easy consequence of Tychonov's theorem that if for a family $\mathfrak{F}$ of finite sets every finite subfamily of it possesses property **G**, then the whole family possesses property **G** too. See e. g. [9].

PROBLEM 8. Let there be given a graph G of power $\aleph_2$. Suppose that every subgraph $G_1 \leq \aleph_1$ of G has chromatic number not greater than $\aleph_0$. Is it then true that the chromatic number of G is not greater than $\aleph_0$?[20]

Now we would like to formulate a problem which does not seem to follow directly from $\boldsymbol{T}(\aleph_2, 1) \to 2$, but which belongs to this class of problems too.

PROBLEM 9. Let there be given a graph G of power $\aleph_2$. Suppose that the edges of every subgraph G_1 of G can be directed so that the number of edges emanating from an arbitrary vertex is finite, provided $G_1 \leq \aleph_1$.

Is it true that the same holds for the graph G?[21]

A positive solution of Problem 9 would follow from the following generalization of TYCHONOV's theorem. (This generalization is probably false, but as far as we know has not yet been disproved.)

PROBLEM 10. Let $\mathfrak{F}$ be a family of *finite* sets, $\overline{\mathfrak{F}} = \aleph_2$, and let $\mathfrak{F} = \{F_\nu\}_{\nu < \omega_2}$ be a well-ordering of type ω_2 of $\mathfrak{F}$. Let $\mathfrak{X}$ denote the Descartes product of the elements of $\mathfrak{F}$, i. e. $\mathfrak{X}$ is the set of all sequences $(x_\nu)_{\nu < \omega_2}$, $x_\nu \in F_\nu$. A subset X of $\mathfrak{X}$ is said to be $\aleph_0$-modified if there exists a set I of ordinal numbers less than ω_2, $\overline{I} \leq \aleph_0$ such that $\overset{1}{x}_\nu = \overset{2}{x}_\nu$ for every $\nu \notin I$ implies that $(\overset{1}{x}_\nu)_{\nu < \omega_2}$ belongs to $\mathfrak{X}$ if and only if $(\overset{2}{x}_\nu)_{\nu < \omega_2}$ belongs to X.

Let $\mathfrak{M}$ be a family of $\aleph_0$-modified subsets of $\mathfrak{X}$ and suppose that the intersection of the elements of every subfamily $\mathfrak{M}'$ of $\mathfrak{M}$ is non-empty, provided $\overline{\mathfrak{M}'} \leq \aleph_1$. Is it true that for an arbitrary family $\mathfrak{M}$ satisfying these conditions $\bigcap_{X \in \mathfrak{M}} X \neq 0$?

9. Further problems. Suppose $p < \aleph_0$.[22] The theorem formulated in the footnote[18] on p. 117 or similar considerations show that to clear up all the problems it would be sufficient to determine the values of the symbols $\mathbf{M}(m, p, q, r) \to \mathbf{B}$, $\mathbf{M}(m, p, r) \to \mathbf{B}(s)$ for finite m's, and so we now suppose that m, p, q, r, s are finite. Obviously, if $r = 1$, then the problems become trivial. So the simplest cases when one can find unsolved problems are $q = 2$, $r = 2$.

[20] It is well known that if every finite subgraph of G has chromatic number not exceeding n, then G has chromatic number not exceeding n. See [10].

[21] As an easy application of TYCHONOV's theorem P. ERDŐS and R. RADO proved the following theorem:

If the edges of every finite subgraph of a given graph G can be directed so that the number of edges emanating from an arbitrary vertex is less than a fixed integer n, then the same is true for the graph G.

[22] T. GALLAI pointed out that interesting and perhaps deep questions can be asked concerning the symbols for p less than $\aleph_0$.

One can ask whether $\mathbf{M}(m, p, 2, 2) \to \mathbf{B}$ is true for a $p > 2$ and for every m. The only non-trivial remark concerning this problem is that

$$\mathbf{M}(7, 3, 2, 2) \not\to \mathbf{B}. \tag{9.1}$$

This is shown by the Steiner triplets for $m = 7$.

The simplest unsolved problem here is

Problem 11. Is it true that

$$\mathbf{M}(m, 4, 2, 2) \to \mathbf{B} \quad \text{holds for every} \quad m?$$

We can not even decide whether there exists an integer p_0 such that

$$\mathbf{M}(m, p_0, 2, 2) \to \mathbf{B} \quad \text{holds for every} \quad m.$$

Remark. The example (9.1) is best-possible in m, i. e. $\mathbf{M}(6, 3, 2, 2) \to \mathbf{B}$ is true and it is interesting that for $m = 6$, $\mathbf{M}(6, 3, 2, 2) \to \mathbf{B}(2)$ is true too. There remain interesting unsolved problems even if we omit the assumption that $\mathfrak{F}$ possesses property $\mathbf{C}(q, r)$ for some q and r.

It is obvious that if m is sufficiently large, then a family $\mathfrak{F}$ with $p(\mathfrak{F}) = p$, $\overline{\mathfrak{F}} = m$ has not to possess property $\mathbf{B}$. Let $m(p)$ denote the least integer m for which such a family exists.

We have

$$m(p) \leqq \binom{2p-1}{p}, \tag{9.2}$$

as it is shown by the subsets taken p at a time of a set having $2p-1$ elements.

More generally, one can ask for the least integers m for which there exists a family $\mathfrak{F}$ ($\overline{\mathfrak{F}} = m$, $p(\mathfrak{F}) = p$) which does not possess property $\mathbf{B}(s)$ where $2 \leqq s \leqq p$. Let $m(p, s)$ denote this integer. (Obviously $m(p, p) = m(p)$.) Similarly as in (9. 2) we have

$$m(p, s) \leqq \binom{p+s-1}{p}. \tag{9.3}$$

(9. 1) shows that the estimations (9. 2) and (9. 3) are far to be best-possible already for $p = 3$. The following problem remains open:

Problem 12. What is the order of magnitude of the functions $m(p)$, $m(p, s)$?

Let us now return to the infinite sets. We would like to raise several new problems, most of which are unsolved, which are all connected to a lesser or greater extent to the ones which we considered so far. To save space we will only outline the partial solutions which we have succeeded in obtaining up to the present.

The first of these problems is the following:

(9.4) Let there be given a family $\mathfrak{F}$ ($\overline{\overline{\mathfrak{F}}} = m$, $p(\mathfrak{F}) = p$) such that every subfamily $\mathfrak{F}'$ of $\mathfrak{F}$ possesses property $\mathbf{B}(r)$, provided that $\overline{\overline{\mathfrak{F}'}} < m$. Under what conditions for the cardinal numbers m, p, r and s does then $\mathfrak{F}$ necessarily possess property $\mathbf{B}(s)$ or property $\mathbf{B}$?

For the sake of brevity we introduce the symbols $\mathbf{S}(m, p, r) \rightarrow \mathbf{B}(s)$, $\mathbf{S}(m, p, r) \rightarrow \mathbf{B}$ ($\mathbf{S}(m, p, r) \not\rightarrow \mathbf{B}(s)$, $\mathbf{S}(m, p, r) \not\rightarrow \mathbf{B}$) to indicate the positive (negative) solutions of the problems, respectively. It is obvious that the problem stated in (9.4) is closely connected with the possible generalizations of TYCHONOV's theorem treated in Section 8. We point out only the simplest and typical problems. A general discussion of this symbol seems to be hopeless at present.

The example given by MILLER cited in Theorem 3 shows, if we assume (*), that

(*)(9.5) $$\mathbf{S}(\aleph_1, \aleph_0, 2) \not\rightarrow \mathbf{B}.$$

This follows from the fact that the system of almost disjoint sets of power $\aleph_1$ constructed by MILLER has the following property: if x is an element of the basic set and $S(x)$ is the union of the sets belonging to the system containing x and F is a set of the system not containing x, then $\overline{\overline{S(x) \cap F}} < \aleph_0$.

Comparing Theorems 8 and 10 we obtain as a corrollary that

(*)(9.6) $$\mathbf{S}(\aleph_2, \aleph_0, 4) \not\rightarrow \mathbf{B}(4).$$

The following problems remain open:

PROBLEM 13. a) $\mathbf{S}(\aleph_2, \aleph_0, 2) \rightarrow \mathbf{B}(2)$ or $\mathbf{S}(\aleph_2, \aleph_0, 2) \not\rightarrow \mathbf{B}$?

b) $\mathbf{S}(\aleph_2, \aleph_0, 4) \rightarrow \mathbf{B}(5)$ or $\mathbf{S}(\aleph_2, \aleph_0, 2) \not\rightarrow \mathbf{B}$?

The following problem concerning the symbol introduced in (9.4) is the simplest one for which our theorems proved so far do not give any information.

PROBLEM 14. Let r be an integer $r \geqq 2$. Is it true that $\mathbf{S}(\aleph_\omega, \aleph_0, r) \rightarrow \mathbf{B}(r)$ holds?

REMARK. It is easy to see that a negative solution of Problem 14 for any r would imply a negative solution of Problem 4.

The second question which arises concerning property $\mathbf{B}$ is the following: Theorem 3 (MILLER's example) assures that there exists a family $\mathfrak{F}$ ($\overline{\overline{\mathfrak{F}}} = 2^{\aleph_0}$, $p(\mathfrak{F}) = \aleph_0$) such that $\mathfrak{F}$ possesses property $\mathbf{C}(2, \aleph_0)$, but it does not possess property $\mathbf{B}$. However, his example is such that $(\mathfrak{F}) = \aleph_0$ and one can ask whether this is an essential restriction.

Concerning this question, using (*), we can prove the following theorem:

(*) (9.7) *There exists a family* $\mathfrak{F}$ $(\overline{\overline{\mathfrak{F}}}=\aleph_1,\ p(\mathfrak{F})=\aleph_0)$ *which possesses property* $\mathbf{C}(2,\aleph_0)$ *such that it does not possess property* **B** *and satisfies the following condition:*

($\triangle$) $(\mathfrak{F}')=\aleph_1$ *for every* $\mathfrak{F}'\subseteq\mathfrak{F}$, $\overline{\overline{\mathfrak{F}'}}=\aleph_1$.

We only outline the construction.

Let S be a set, $\overline{\overline{S}}=\aleph_1$. Applying Lemma 3 stated in Section 4 we obtain that there exists a system $\mathfrak{S}$ of subsets of S satisfying the following conditions:

(1) $p(\mathfrak{S})=\aleph_0$, $\overline{\overline{\mathfrak{S}}}=\aleph_1$.

(2) $\mathfrak{S}$ possesses property $C(2,\aleph_0)$.

(3) For an arbitrary $S'\subseteq S$ $(\overline{\overline{S'}}=\aleph_1)$ there exists an $A\in\mathfrak{S}$ such that $A\subseteq S'$.

Let $\mathfrak{S}=\{A_\nu\}_{\nu<\omega_1}$ and $S=\{x_\mu\}_{\mu<\omega_1}$ be well-orderings of type ω_1 of the sets $\mathfrak{S}$ and S, respectively.

Let $\mathfrak{S}_\nu$ be a system of subsets of A_ν for which $p(\mathfrak{S}_\nu)=\aleph_0$, $\overline{\overline{\mathfrak{S}_\nu}}=\aleph_1$, further let $\mathfrak{S}_\nu$ possess property $\mathbf{C}(2,\aleph_0)$. Let $\mathfrak{S}_\nu=\{B_\mu^\nu\}_{\mu<\omega_1}$ be a well-ordering of type ω_1 of the set $\mathfrak{S}_\nu$ for every $\nu<\omega_1$. It is obvious that one can define a monotone increasing sequence $\{\mu_\nu\}_{\nu<\omega_1}$ of type ω_1 of ordinal numbers less than ω_1 such that $\mu_\nu>\mu'$ for every $x_{\mu'}\in A_\nu$ (hence for every $x_{\mu'}\in B_\mu^\nu$ for every $\mu<\omega_1$).

Put $C_\mu^\nu=B_\mu^\nu\cup\{x_{\mu_\nu+\mu}\}$ and $\mathfrak{F}=\{C_\mu^\nu\}_{\nu<\omega_1,\,\mu<\omega_1}$. It is obvious from (1) and (2) that $\overline{\overline{\mathfrak{F}}}=\aleph_1$, $p(\mathfrak{F})=\aleph_0$ and $\mathfrak{F}$ possesses property $\mathbf{C}(2,\aleph_0)$. The fact that $\mathfrak{F}$ does not possess property **B** follows from the property of $\mathfrak{S}$ stated in (3) (taking into consideration that a set which intersects every element of $\mathfrak{F}$ has to be of power $\aleph_1$). Finally, it is easy to verify that if $\overline{\overline{\mathfrak{F}'}}=\aleph_1$, then $(\overline{\mathfrak{F}'})=\aleph_1$ for every $\mathfrak{F}'\subseteq\mathfrak{F}$, since if $\overline{\overline{\mathfrak{F}'}}=\aleph_1$, then $\mathfrak{F}'$ either contains $\aleph_1$ C_μ^ν's with the same ν or $\aleph_1$ C_μ^ν's with pairwise different ν's.

The following refinement of the problem solved in (9.7) seems to be interesting. Let us say that the set X is almost contained in Y if $Y-X$ is finite.

Problem 15. Let $\mathfrak{F}$ be a family $(p(\mathfrak{F})=\aleph_0,\ \overline{\overline{\mathfrak{F}}}=\aleph_1)$ such that $\mathfrak{F}$ possesses property $\mathbf{C}(2,\aleph_0)$ and suppose that (instead of ($\triangle$)) it possesses the following property:

At most $\aleph_0$ sets belonging to $\mathfrak{F}$ are almost contained in a denumerable set.

Does such a family $\mathfrak{F}$ necessarily possess property **B**?

The answer is probably negative to this question too, but we can not disprove it even if we omit the assumption that $\mathfrak{F}$ consists of almost disjoint sets.

The following question is connected with Problem 3 (namely a positive solution of it would imply a positive solution of Problem 3b)):

PROBLEM 16. Put $S = \{\nu\}_{\nu<\omega_{\omega+1}}$ $(\bar{\bar{S}} = \aleph_{\omega+1})$.

Let S_ν denote the set $\{\mu\}_{\mu<\nu}$ for every $\nu<\omega_{\omega+1}$. Then $\bar{\bar{S}}_\nu \leqq \aleph_\omega$, and so one can define a splitting of S_ν onto the sum of $\aleph_0$ disjoint sets such that

$$S_\nu = \bigcup_{n<\omega} S_n^\nu \quad \text{and} \quad \bar{\bar{S}}_n^\nu < \aleph_\omega \quad \text{for every} \quad \nu<\omega_{\omega+1}.$$

Is it possible to define the sets S_n^ν in such a way that for every $\nu<\omega_{\omega+1}$ of the second kind which is not cofinal with ω, there exists a monotone increasing sequence $\{\nu_\tau\}_{\tau<\varphi}$ of type φ of ordinal numbers less than ν cofinal with ν and such that $S_n^{\nu_\tau} \subseteq S_n^{\nu_{\tau'}}$ for every n and for every $\tau<\tau'<\varphi$?

A similar but simpler problem is the following one:

PROBLEM 17. Let S be the set of ordinal numbers less than ω_1. Is it possible to define a function $f(\nu)$ on S such that $f(\nu) \in S$, $f(\nu)<\nu$ for every $\nu<\omega_1$ which has the following property: If $\nu<\omega_1$ and ν is of the second kind, then there exists a sequence $\{\nu_n\}_{n<\omega}$ of type ω of ordinal numbers less than ν such that $\nu_n \to \nu$ and $f(\nu_{n+1}) = \nu_n$ for $n = 0, 1, 2, \ldots$. This problem is interesting in itself and seems to be very difficult.

The positive solution of the following problem would imply a negative solution of an immediate generalization of Problem 9, namely it would assure the existence of a graph G of power $\aleph_{\omega+1}$ the edges of every subgraph of power $\aleph_1$ of which can be directed so that the number of edges emanating from a vertex should be finite, but the whole graph can not be directed in such a way.

PROBLEM 18. Let S be a set of power $\aleph_\omega$. Does there exist a family $\mathfrak{F}$ such that $(\mathfrak{F}) \subseteq S$, $\bar{\bar{\mathfrak{F}}} = \aleph_{\omega+1}$, $p(\mathfrak{F}) = \aleph_0$, and which has the following property:

(1) If $S' \subseteq S$, $\bar{\bar{S}}' < \aleph_0$, then there exist at most $\aleph_0$ sets F belonging to the family such that $\overline{\overline{F \cap S'}} = \aleph_0$.

REMARK. On the one hand, we can not disprove Problem 18 even if we require that $\mathfrak{F}$ should possess property $\mathbf{C}(2, \aleph_0)$, on the other hand we can not prove it if we require only that $\mathfrak{F}$ should possess the following weaker property instead of (1):

Every $S' \subseteq S$ $(\bar{\bar{S}}' = \aleph_0)$ contains at most $\aleph_0$ elements of the family.

We construct the graph G mentioned above as follows: Suppose that the family $\mathfrak{F}$ and the set S satisfy the requirement of Problem 18. Let the set of vertices of G be $\mathfrak{F} \cup S$. The edges are the pairs (F, x) where $F \in \mathfrak{F}$ and $x \in F$. It is easy to see that G has the property required.

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Added in proof (MARCH 3, 1961). The manuscript of this paper had been written before the authors knew that A. TARSKI has disproved the hypothesis (**). (See A. TARSKI, Some problems and results relevant to the foundations of set theory, *Proceedings of the International Congress for Logic, Methodology and Philosophy of Science* (Stanford, 1960).)

Thus we have no arguments to prove our Theorem 12 proved with the help of this hypothesis. It seems that the theorem is false at least for the inaccessible cardinals m which are strongly incompact.

It is obvious that the discussion of the unsolved problems concerning the symbol $\boldsymbol{T}(m, \lambda) \rightarrow \varkappa$ has to be changed in some places knowing the new result of A. TARSKI.

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SOME REMARKS AND PROBLEMS ON THE COLOURING OF INFINITE GRAPHS AND THE THEOREM OF KURATOWSKI[1]

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(*Presented by* G. HAJÓS)

1. We consider the following propositions:

T. The topological product of any number of bicompact Hausdorff spaces is bicompact.[2]

T*. The topological product of any number of non-empty bicompact Hausdorff spaces is non-empty and bicompact.

I. In every Boolean algebra A there is a maximal ideal different from A.

R. Every Boolean algebra is isomorphic to a field of sets.

M. Every consistent elementary theory has a model.[3]

T_n. The topological product af any number of Hausdorff spaces, each having exactly n points, is non-empty and bicompact.[4]

S_n. Let M be any set of disjoint sets each having exactly n elements and $R(x, y)$ is a symmetric relation defined between the elements belonging to different elements of M. Suppose that for any finite set $F \subseteq M$ there exists an $f \in \mathop{\mathsf{P}}\limits_{X \in F} X$ such that $R(f(X_1), f(X_2))$ holds for any $X_1, X_2 \in F$. Then there exists an $f \in \mathop{\mathsf{P}}\limits_{X \in M} X$ such that $R(f(X_1), f(X_2))$ holds for any $X_1, X_2 \in M$.[5]

P_n. Every graph, each finite subgraph of which can be coloured with n colours, can be coloured with n colours.[6]

C_n. The Cartesian product of any number of sets, each having exactly n elements, is non-empty.

It is known that the axiom of choice implies each of the above propo-

[1] This is a lecture delivered on the Colloquium on the Theory of Graphs in Dobogókő, 22 October 1959.

[2] Here and farther *any number* means any positive finite or infinite cardinal number.

[3] We do not suppose that the number of symbols and statements of the theories is denumerable.

[4] Here and farther n is running over positive integers.

[5] P denotes the Cartesian product operator.

[6] A colouring of a graph with n colours is a partition of the set of vertices into n classes such that no two vertices in one class are joint by an edge.

sitions, but the following logical relations can be proved without the use o this axiom:

(1) $T \leftrightarrow T^* \leftrightarrow I \leftrightarrow R \leftrightarrow M \leftrightarrow T_m \leftrightarrow S_n$ for $m=2,3,\ldots,$ $n=4,5,\ldots;$

(2) $S_4 \rightarrow S_3 \rightarrow S_2;$

(3) $S_n \rightarrow P_n \rightarrow C_n$ for $n=2,3,\ldots;$

(4) $P_{n+1} \rightarrow P_n$ for $n=2,3,\ldots;$

(5) $P_2 \leftrightarrow C_2.$

It would be interesting to know any further implication between these propositions. Some implications and independences between the propositions C_n are known, e. g. $C_2 \leftrightarrow C_4$, $C_{mn} \rightarrow C_m$ and others (see [8], [10], [11]). The equivalences (1) and implications (2) are proved in the papers [4], [5], [6], [7]. Other interesting propositions which may be added to the equivalences (1) are given in [9].

Let us prove (3), (4) and (5):

$S_n \rightarrow P_n$ is obvious (compare the proof of P_n given in [2]).

$P_n \rightarrow C_n$. Let K be a set of disjoint n-element sets. We treat $\bigcup_{X \in K} X$ as a set of vertices of a graph, two vertices being joint if and only if they belong to the same X. By P_n it is easy to see that this graph can be coloured with n colours. Take all the vertices of one colour, this clearly defines a selection from K as required in C_n.

$P_{n+1} \rightarrow P_n$. Let G be a graph each finite subgraph of which can be coloured with n colours. We add a new vertex and join it to all vertices of G. Using P_{n+1} we easily see that the new graph can be coloured with $n+1$ colours. Removing the additional vertex we obtain n-colourings of G as needed in P_n.

$P_2 \leftrightarrow C_2$. Owing to (3) it remains to prove $C_2 \rightarrow P_2$. If G is a connected graph, each finite subgraph of which can be coloured with 2 colours, then it is easy to see that, putting two vertices in the same class if and only if there exists a path from one to the other with an odd number of edges, we obtain a two-colouring of G. Now if G is not connected, using C_2 we select one of these classes for each component of G. We consider the partition of the vertices of G into 2 classes: the union of the selected classes and the remaining vertices. It is easy to see that it is a two-colouring of G.

Remark (due to C. Ryll-Nardzewski). The proposition P_n restricted to denumerable graphs can be proved without using the axiom of choice.

2. We consider the following properties of a graph G (by a graph we mean here a one-dimensional simplicial complex with the natural topology,

we do not suppose that it is locally finite and the cardinality of the set of vertices of G is arbitrary):

(i) G does not contain topologically any one of KURATOWSKI's two graphs (Fig. 1).

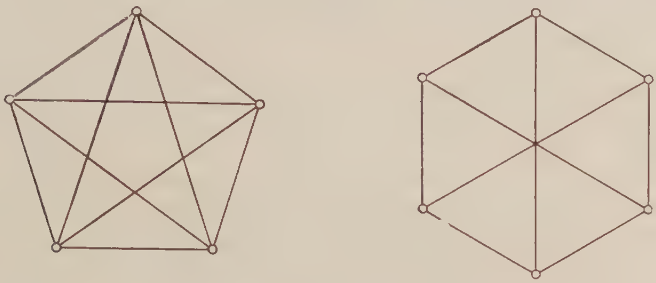

Fig. 1

(ii) Every finite subgraph of G is homeomorphically imbeddable in the plane R^2.

(iii) There exists a system of homeomorphisms $\{h_F(x)\}$ where F runs over all finite subgraphs of G such that h_F maps homeomorphically F into R^2 and for any F_1 and F_2

$$(*) \qquad h_{F_1}|F_1 \cap F_2 \text{ is homotopical to } h_{F_2}|F_1 \cap F_2.$$ [7]

(iv) One can define for every circuit C of G a partition of the set $|G| \setminus |C|$ [8] into two classes Int (C), Ext (C) such that two vertices belonging to different classes are not joint by an edge and

$$\text{if } |C_1| \subset |C_2| \cup \text{Int}(C_2), \text{ then } \text{Int}(C_1) \subset |C_2| \cup \text{Int}(C_2);$$
$$\text{if } |C_1| \subset |C_2| \cup \text{Ext}(C_2), \text{ then } \text{Ext}(C_1) \subset |C_2| \cup \text{Ext}(C_2).$$

THEOREM. *The properties* (i), (ii), (iii), (iv) *are equivalent.*

PROOF. (i) $\rightarrow$ (ii) by the well-known theorem of KURATOWSKI [3].

(ii) $\rightarrow$ (iii). We denote by S_F the set of homotopy types of homeomorphical applications of F into R^2 (F runs over the finite subgraphs of G). S_F is finite; we treat it as a discrete topological space. By the statement T (Section 1 of this paper) the topological product $\mathop{\mathsf{P}}_F S_F$ is bicompact.

For any $t_1 \in S_{F_1}$ and $t_2 \in S_{F_2}$ we put $t_1 \sim t_2$ if and only if $(*)$ holds for some h_{F_1} of type t_1 and h_{F_2} of type t_2. Let $F_1, \ldots, F_m$ be any finite set of

[7] $f|X$ denotes the mapping f with domain restricted to X.

[8] $|H|$ denotes the set of vertices of the graph H. $\setminus$ denotes the set-theoretical difference.

finite subgraphs of G. We put $K_{F_1,\dots,F_m} = \{f : f \in \mathop{\mathsf{P}}_F S_F,\ f(F_i) \sim f(F_j)$ for $i, j = 1, \dots, m\}$. Of course, the sets $K_{F_1,\dots,F_m}$ are closed subsets of $\mathop{\mathsf{P}}_F S_F$. They are also non-empty, since if F is a finite subgraph of G such that $F_1, \dots, F_m$ are subgraphs of F and h_F is a homeomorphism $h_F : F \to R^2$ (it exists by (ii)), then one can take for $f \in K_{F_1,\dots,F_m}$ any function $f \in \mathop{\mathsf{P}}_F S_F$ such that $f(F_i)$ is the homotopy type of $h_F|F_i$. The finite intersections of the sets $K_{F_1,\dots,F_m}$ are also non-empty, since

$$K_{F_1^{(1)},\dots,F_m^{(1)}} \cap K_{F_1^{(2)},\dots,F_n^{(2)}} \supset K_{F_1^{(1)},\dots,F_m^{(1)},\,F_1^{(2)},\dots,F_n^{(2)}}.$$

It follows that there exists an f_0 such that

$$f_0 \in \bigcap_{m=1}^{\infty} \bigcap_{F_1,\dots,F_m} K_{F_1,\dots,F_m}$$

and clearly any system $\{h_F\}$, such that the homotopy type of h_F is $f_0(F)$ satisfies (iii); q. e. d.

(iii) $\to$ (iv). A system $\{h_F\}$ being given, for every circuit C and every vertex $v \in |G| \setminus |C|$ we put $v \in \operatorname{Int}(C)$ if the homeomorphism $h_{C \cup (v)}$ maps v inside the domain bounded by the image of C and $v \in \operatorname{Ext}(C)$ in the other case. It is easy to verify that our definition satisfies (iv).

(iv) $\to$ (i). Clearly a subgraph of a graph satisfying (iv) satisfies (iv) One can prove by a direct verification that no one of the Kuratowski graphs satisfies (iv); and our implication follows.

COROLLARY. (DIRAC and SCHUSTER [1].) *A denumerable graph satisfying* (i) *has a continuous* 1—1 *mapping into* R^2.

PROOF. By the theorem the graph satisfies (iii) and one can construct the mapping by an easy induction.

REMARK. The equivalence (ii) $\leftrightarrow$ (iii) remains valid if one replaces in these statements R^2 by any bicompact 2-manifold.

PROBLEM. Does there exist a finite set of finite graphs such that any finite graph G can not be homeomorphically imbedded in a given bicompact 2-manifold (e. g. the projective plane) if and only if G contains a subgraph homeomorphic to one of them?

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MAXIMUM-MINIMUM SÄTZE UND VERALLGEMEINERTE FAKTOREN VON GRAPHEN

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Einleitung

Von D. KÖNIG stammt der folgende Satz ([5], S. 233, Satz 14):[1]

SATZ (1) (KÖNIG). *In einem paaren Graphen*[2] *ist die maximale Anzahl der unabhängigen Kanten gleich der minimalen Anzahl der trennenden Punkte.*

Die Kanten $x_1, \ldots, x_m$ heißen unabhängig, wenn entweder $m=1$ ist, oder wenn je zwei von ihnen keinen gemeinsamen Punkt enthalten. Die Punkte $X_1, \ldots, X_n$ heißen trennend, wenn jede Kante des Graphen einen dieser Punkte enthält. Die Behauptung des Satzes (1) ist für nichtpaare Graphen im allgemeinen nicht richtig. Es entsteht daher das Problem: durch geeignete Modifizierung der vorkommenden Größen eine auf beliebige Graphen geltende Verallgemeinerung zu finden. Wir wollen nun in dieser Arbeit zeigen, daß man die minimale Anzahl der trennenden Punkte (kurz: $p_{\min}$) durch eine solche Größe ersetzen kann, die im Falle paarer Graphen mit $p_{\min}$ zusammenfällt, und mit welcher die Behauptung des Satzes (1) für beliebige Graphen richtig ist. Diese Größe wird sich als das Minimum der geeignet bestimmten Werte von gewissen Gewichtssystemen ergeben. Weitergehend werden wir den Satz (1) auch in anderen Richtungen verallgemeinern. Im Satze (1) spielen nur Systeme von Kanten eine Rolle. Statt dieser werden wir aus Bogen, d. h. aus Wegen und Schlingen[3] zusammengesetzte Systeme in Betracht ziehen. Um den Begriff „unabhängig" auszudehnen, ordnen wir jedem Punkte X des Graphen zwei nichtnegative ganze Zahlen: eine Einlaufskapazität $\varkappa(X)$ und eine Durchgangskapazität $\varkappa'(X)$ zu. Ein aus Bogen bestehendes System H soll bezüglich $\varkappa$ und $\varkappa'$ aufnehmbar heißen, wenn je zwei Bogen von H keine gemeinsame Kante enthalten, in jedem Punkte X

[1] Wir sagen statt Knotenpunkte kurz nur Punkte.

[2] Ein Graph heißt paarer, wenn die Menge der Punkte des Graphen so in zwei Teilmengen zerlegt werden kann, daß jede Kante des Graphen zwei Punkte von verschiedenen Teilmengen verbindet.

[3] Eine Schlinge ist ein Kreis mit einem ausgezeichneten Punkte X des Kreises. Wir betrachten X als einen zweifachen Randpunkt der Schlinge.

höchstens $\varkappa(X)$ Randpunkte der Bogen von H fallen, und durch X höchstens $\varkappa'(X)$ Bogen von H gehen. Ersetzt man dann im Satze (1) die maximale Anzahl der unabhängigen Kanten durch die maximale Anzahl der aufnehmbaren Bogen, die Größe $p_{\min}$ wieder durch das Minimum der „Werte" von gewissen Gewichtssystemen, so entsteht ein allgemeiner Maximum-Minimum Satz, der Hauptsatz unserer Arbeit (Satz (3.3)). Aus diesem Satze ziehen wir mehrere Folgerungen. Unter anderem geben wir zwei, dem Mengerschen n-Kettensatz ([6], S. 222) ähnliche Trennungssätze (Sätze (13.2) und (13.4)). Wir leiten gleichfalls aus unserem Hauptsatze eine Reihe von Faktorisationssätzen ab. Ist jedem Punkte X des Graphen nur eine nichtnegative ganze Zahl $\varkappa(X)$ zugeordnet, so versteht man unter einem $\varkappa$-Faktor eine Menge von Kanten mit der Eigenschaft, daß zu jedem Punkte X genau $\varkappa(X)$ Kanten der Menge inzident sind (s. [10], S. 316). Sind zu jedem Punkte X zwei nichtnegative ganze Zahlen $\varkappa(X)$ und $\varkappa'(X)$ zugeordnet, so wollen wir unter einem verallgemeinerten Faktor ein solches, bezüglich $\varkappa$ und $\varkappa'$ aufnehmbares Bogensystem H verstehen, bei welchem in jeden Punkt X des Graphen genau $\varkappa(X)$ Randpunkte von H fallen. Es werden nun allgemeine notwendige und hinreichende — sowie im Falle spezieller Graphen und Kapazitäten einfache hinreichende — Bedingungen der Existenz von verallgemeinerten Faktoren angegeben (§ 14—16). Diese enthalten als Spezialfälle mehrere Sätze über $\varkappa$-Faktoren von PETERSEN, BAEBLER, TUTTE, BELCK und ORE.

Wir leiten unsere Sätze nur für solche Graphen ab, die keine Kantenschlingen, d. h. Kanten mit zusammenfallenden Endpunkten enthalten. Es bietet aber keine Schwierigkeit, unsere Ergebnisse auch auf Graphen mit Kantenschlingen auszubreiten.

Den Hauptsatz beweisen wir mit einer geeigneten Verallgemeinerung der Methode der alternierenden Züge (s. [3], [4], [10], [11]). Den Spezialfall, wo sämtliche $\varkappa'$-Werte verschwinden, kann man wesentlich kürzer behandeln. Es scheint wahrscheinlich zu sein, daß der allgemeine Fall auf diesen Spezialfall zurückführbar ist. Auf diese Weise könnte man zu einem einfacheren Beweis des Hauptsatzes gelangen.

Die vorliegende Arbeit ist in vier Abschnitte geteilt. Im ersten Abschnitt (§ 1—3) geben wir die Erklärung der nötigen Grundbegriffe und formulieren den Hauptsatz. Der zweite (§ 4—10) enthält den Beweis des Hauptsatzes. Im dritten (§ 11—13) untersuchen wir einige Spezialfälle des Hauptsatzes und leiten die erwähnten Trennungssätze ab. Der vierte Abschnitt (§ 14—16) enthält die Faktorisationssätze.

Um die weniger wichtigen Behauptungen von den eigentlichen Sätzen zu unterscheiden, lassen wir die Benennung „Satz" bei den ersteren weg.

I. FORMULIERUNG DES HAUPTSATZES

§ 1. Grundbegriffe

(1.1) Der ungerichtete, endliche Graph Γ ist durch zwei elementenfremde endliche Mengen, durch die Menge Φ_Γ der „Punkte" und die Menge Ψ_Γ der „Kanten" sowie durch eine Inzidenzvorschrift I_Γ gegeben. I_Γ gibt an, ob ein beliebiger Punkt von Φ_Γ und eine beliebige Kante von Ψ_Γ *inzident* sind oder nicht, und diese Vorschrift genügt nur der einen Bedingung, daß zu jeder Kante genau zwei (verschiedene) Punkte inzident sind (s. [5]). Wir bezeichnen einen Punkt bzw. eine Kante immer mit X bzw. mit x, eventuell auch mit Indizes oder anderen Zeichen versehen. Ist nach I_Γ die Kante x zu den verschiedenen Punkten X_1 und X_2 inzident, so sagen wir: x *verbindet* die Punkte X_1 und X_2, X_1 und X_2 sind die Punkte oder *Randpunkte* von x, x ist eine *X_1X_2-Kante.* Von den Punkten und Kanten von Γ werden wir auch sagen, daß sie *in* oder *auf* Γ liegen.

Sind Φ_Γ und Ψ_Γ beide leer, so heißt der Graph Γ *leer*. $\Gamma' \subseteq \Gamma$ soll bedeuten: der Graph Γ' ist ein *Teilgraph* von Γ, d. h. es gilt $\Phi_{\Gamma'} \subseteq \Phi_\Gamma$, $\Psi_{\Gamma'} \subseteq \Psi_\Gamma$ und ein Punkt und eine Kante von Γ' sind in Γ' dann und nur dann inzident, wenn sie in Γ inzident sind. Ist $\Gamma' \subseteq \Gamma$ und $\Gamma' \neq \Gamma$, so heißt Γ' ein *echter* Teilgraph von Γ, und wir schreiben $\Gamma' \subset \Gamma$. Ein Teilgraph Γ' von Γ ist durch Angabe der Mengen $\Phi_{\Gamma'}$ und $\Psi_{\Gamma'}$ eindeutig bestimmt.

Ist $\Gamma' \subset \Gamma$ und liegt nur der eine Randpunkt der Kante x von Γ in Γ', so sagen wir, daß x den Graphen Γ' *berührt*, und der nicht zu Γ' gehörige Randpunkt von x heißt der *äußere* Punkt von x (bezüglich Γ').

Ist M eine beliebige endliche Menge, so bezeichnen wir mit $\nu(M)$ die Anzahl der Elemente von M. Die leere Menge bezeichnen wir mit $\varnothing$.

Ist E eine beliebige Teilmenge von Φ_Γ, so nennen wir die Punkte von E kurz *E-Punkte*. Ist $\varphi(X)$ eine beliebige in Φ_Γ definierte Funktion, so soll $\varphi(E)$ im Falle $E \neq \varnothing$ den Wert $\sum\limits_{X \in E} \varphi(X)$, im Falle $E = \varnothing$ den Wert Null bedeuten.

Es sei $E \subseteq \Phi_\Gamma$ und $F \subseteq \Phi_\Gamma$. Wir nennen jene Kanten, bei denen ein Randpunkt ein E-Punkt und der andere ein F-Punkt ist, eine *EF-Kante*. Die Anzahl der EF-Kanten von Γ bezeichnen wir mit $\nu_\Gamma(E, F)$. Es gilt also z. B. $\nu_\Gamma(\Phi_\Gamma, \Phi_\Gamma) = \nu(\Psi_\Gamma)$. Ist eine der Mengen E und F leer, so sei $\nu_\Gamma(E, F) = 0$. Ist $E = \{X\}$, so schreiben wir statt EF-Kanten bzw. statt $\nu_\Gamma(E, F)$ auch XF-Kanten und $\nu_\Gamma(X, F)$. Die Zahl $\nu_\Gamma(X', X)$ gibt die Anzahl der XX'-Kanten von Γ an.

Das Zeichen $\varrho_\Gamma(X)$ soll den *Grad* von X in Γ, d. h. die Anzahl der

zu X inzidenten Kanten von Γ bedeuten. Ist $\varrho_\Gamma(X)=0$, so heißt X ein *isolierter* Punkt von Γ.

Ist $E\subseteq\Phi_\Gamma$, so soll $[E]_\Gamma$ denjenigen Teilgraphen von Γ bezeichnen, der sämtliche E-Punkte und EE-Kanten von Γ und nur diese enthält. (Ist also $E=\emptyset$, so ist $[E]_\Gamma$ der leere Graph.)

Wir machen folgende Vereinbarungen: Γ soll immer einen endlichen ungerichteten Graphen bezeichnen, ferner wollen wir von den Zeichen bzw. Begriffen, die sich auf Γ beziehen, den Index Γ bzw. die Ausdrücke „von Γ", „in Γ" usw. im allgemeinen weglassen. Diese beziehen sich also — wenn anders nicht gesagt wird — immer auf den mit Γ bezeichneten Graphen. So bedeutet z. B. Φ die Menge Φ_Γ, und der Ausdruck „für jeden X" den Ausdruck „für jeden X von Γ".

Ist $E\subseteq\Phi$, so setzen wir $\overline{E}=\Phi-E$.

(1.2) Sind $X_0,\ldots,X_n$ $(n\geqq 1)$[4] (nicht unbedingt verschiedene) Punkte und $x_1,\ldots,x_n$ *verschiedene* Kanten von Γ, sind ferner X_{i-1} und X_i die beiden Randpunkte von x_i $(i=1,\ldots,n)$, so heißt die Folge

$$p=(X_0x_1X_1\ldots X_{n-1}x_nX_n)$$

ein *Zug* (von Γ). Kommt der Punkt X in der Folge $X_0,\ldots,X_n$ genau m-mal vor $(m\geqq 1)$, so heißt X ein *m-facher* Punkt von p. Im Falle $m=1$ bzw. $m>1$ sprechen wir auch von *einfachen* bzw. *mehrfachen* Punkten von p (wir sagen auch, daß X in p einfach bzw. m-fach ist). X_0 ist der *Anfangspunkt*, X_n der *Endpunkt* von p, beide heißen die *Randpunkte* von p. Ähnlicherweise bezeichnen wir die Kanten x_1 und x_n. Die Anzahl der Kanten von p heißt die *Länge* von p.

Wir heben hervor, daß mit einem Zuge p eine bestimmte „Durchlaufsrichtung" verbunden ist, die jeder Kante von p eine *eindeutig* bestimmte Reihennummer zuteilt. (Das gleiche gilt für die Punkte im allgemeinen nicht!)

(1.3) Ist $p=(X_0x_1X_1\ldots X_{n-1}x_nX_n)$ ein Zug von Γ, so bezeichne

$$[p]=[X_0x_1X_1\ldots X_{n-1}x_nX_n]$$

denjenigen Teilgraphen von Γ, der sämtliche Punkte und Kanten von p, und nur diese, enthält. Sind $X_0,\ldots,X_n$ verschieden, so heißt $[p]$ ein *Weg*. X_0 und X_n sind die Randpunkte des Weges. Sind $X_0,\ldots,X_{n-1}$ $(n\geqq 2)$ verschieden und $X_n=X_1$, so heißt $[p]$ ein *Kreis*. Ein Kreis mit einem ausgezeichneten Punkt X' des Kreises heißt eine *Schlinge*. Wir nennen X' den Randpunkt der Schlinge, bzw. wir sagen, daß in X' *zwei* Randpunkte der

[4] Sind m und n ganze Zahlen und ist $m<n$, so bedeutet $(a_m,\ldots,a_n)$ diejenige Folge, die dadurch zustande kommt, daß der Index i in a_i jede ganze Zahl von m bis n durchläuft. Im Falle $m=n$ bedeutet $(a_m,\ldots,a_n)$ das einzige Element a_m.

Schlinge fallen. Wege und Schlingen heißen gemeinsam *Bogen*. Die von den Randpunkten verschiedenen Punkte eines Bogens heißen *innere* Punkte des Bogens. Ist h ein Bogen, so bedeutet die Gleichung

$$h = [X_0 x_1 X_1 \cdots X_{n-1} x_n X_n],$$

daß $p = (X_0 x_1 X_1 \cdots X_{n-1} x_n X_n)$ ein Zug ist und h und p dieselben Punkte, Kanten und Randpunkte besitzen.

Den Weg $[X_0 x_1 X_1]$ kann man mit der Kante x_1 selbst identifizieren. Wollen wir eine Kante gleich mit ihren Randpunkten angeben, so schreiben wir sie in der Form $[X_0 x_1 X_1]$.

(1.4) Das Zeichen $[h, X]$ bzw. $|h, X|$ soll angeben, wie viele Randpunkte bzw. inneren Punkte des Bogens h in X fallen. $[h, X]$ bzw. $|h, X|$ können nur die Werte 0, 1 oder 2 bzw. 0 oder 1 annehmen.

(1.5) Γ heißt *zusammenhängend*, wenn er höchstens einen Punkt enthält oder wenn zu je zwei Punkten von Γ ein Weg von Γ existiert, der beide Punkte enthält. Die maximalen zusammenhängenden Teilgraphen von Γ sind die *Komponenten* von Γ. Ist Γ nicht zusammenhängend, so besteht er aus mehreren nichtleeren Komponenten.

Ist $E \subseteq \Phi$, so nennen wir die Komponenten des Graphen $[E]$ die *E-Komponenten* von Γ und bezeichnen diese mit $[E_1], \ldots, [E_m]$ $(m \geqq 1)$. Ferner soll hier E_i $(1 \leqq i \leqq m)$ die Menge der Punkte von $[E_i]$ bedeuten. Ist $E = \varnothing$, so ist $m = 1$, $E_1 = \varnothing$ und $[E_1]$ der leere Graph.

§ 2. Bogen- und Gewichtssysteme

Von hier an bezeichne Γ durch die ganze Arbeit immer einen *nichtleeren* Graphen.

(2.1) Sind $h_1, \ldots, h_n$ Bogen von Γ, so heißt die Folge $H = (h_1, \ldots, h_n)$ ein *Bogensystem* von Γ. Wir betrachten zwei Systeme, die sich nur in der Reihenfolge ihrer Glieder unterscheiden, als identisch. Die leere Folge betrachten wir auch als ein Bogensystem. $H = 0$ soll ausdrücken, daß H leer ist.

$\nu(H)$ bezeichne die Anzahl der Glieder von H. Nach (1.4) gibt im Falle $H \neq 0$

$$[H, X] = \sum_{i=1}^{n} [h_i, X] \quad \text{bzw.} \quad |H, X| = \sum_{i=1}^{n} |h_i, X|$$

an, wie viele Randpunkte bzw. inneren Punkte der Bogen von H insgesamt in X fallen. Ist $H = 0$, so sei $[H, X] = |H, X| = 0$. Es gilt offensichtlich für jedes System H von Γ

$$\sum_{X \in \Phi} [H, X] = 2\nu(H). \tag{1}$$

(2. 2) Ein zu Γ gehöriges *Gewichtssystem* q kommt dadurch zustande, daß wir zu jedem Punkt und zu jeder Kante von Γ eines der Gewichte 0, 1 oder 1/2 zuordnen. $q(X)$ bzw. $q(x)$ soll das Gewicht bezeichnen, welches in q zu X bzw. x gehört.

Wir sagen, daß der Bogen $h=[X_0 x_1 X_1 \ldots X_{n-1} x_n X_n]$ durch das Gewichtssystem q *gefüllt* ist, wenn einer der folgenden Fälle besteht: (1) Es gibt ein i $(0 \leqq i \leqq n)$ mit $q(X_i)=1$. (2) Es gibt ein i $(1 \leqq i \leqq n)$ mit $q(x_i)=1$. (3) Es gilt $q(X_0)=q(X_n)=1/2$. (4) Es ist $q(X_0)=q(x_n)=1/2$ oder $q(x_1)= =q(X_n)=1/2$. (5) Es gilt $q(x_1)=q(x_n)=1/2$ $(n \geqq 2)$.

Wir wollen nur solche Systeme q betrachten, die jeden Bogen von Γ füllen. Man kann leicht einsehen: füllt q jede Kante, so füllt es auch jeden Bogen. Bezeichnet man mit A, B bzw. C die Menge der Punkte, die in q das Gewicht 0, 1 bzw. 1/2 bekommen, so muß in q, falls q jede Kante füllt, jede AA-Kante das Gewicht 1, jede AC-Kante das Gewicht 1 2 erhalten. Ferner ist es klar, daß diese Kantengewichte zusammen mit den Punktgewichten schon zu der Füllung sämtlicher Kanten genügen. Deshalb werden wir im folgenden nur solche Systeme betrachten, die außer den obenerwähnten keine von Null verschiedenen Gewichte enthalten. Diese Systeme sind durch die Mengen A, B und C eindeutig bestimmt. Zusammenfassend: Jedes von uns betrachtete sog. *füllende Gewichtssystem* q geben wir durch eine *geordnete Zerlegung* der Menge Φ in drei Teilmengen an, d. h. wir geben drei beliebige, paarweise fremde Teilmengen A, B und C von Φ mit $A \cup B \cup C=\Phi$ sowie eine bestimmte Reihenfolge A, B, C dieser Mengen an. Jeder A-, B- bzw. C-Punkt soll in q das Gewicht 0, 1 bzw. 1/2, jede AA- bzw. AC-Kante das Gewicht 1 bzw. 1/2, jede übrige Kante das Gewicht 0 erhalten. Diese Bestimmung von q werden wir kurz durch

$$q=q(A, B, C)$$

ausdrücken.

Wir bezeichnen die Menge der so definierten Gewichtssysteme mit Q. Q ist nichtleer.

§ 3. Aufnehmbare Bogensysteme. Der Wert eines Gewichtssystems

(3. 1) Es seien $\varkappa(X)$ und $\varkappa'(X)$ zwei auf der Menge Φ der Punkte von Γ definierte Funktionen, die nur nichtnegative ganze Werte annehmen. Wir wollen diese „Kapazitätsfunktionen" festhalten und die Werte von $\varkappa(X)$ als Einlaufskapazitäten, diejenige von $\varkappa'(X)$ als Durchgangskapazitäten betrachten. Ein Bogensystem H von Γ nennen wir in bezug auf $\varkappa$ und $\varkappa'$ *aufnehmbar*, wenn H folgenden Bedingungen genügt:

(a) Enthält H mehr als einen Bogen, so haben diese Bogen paarweise keine gemeinsame Kante.

(b) Für jedes X gilt $[H, X] \leq \varkappa(X)$ und $|H, X| \leq \varkappa'(X)$.

Wir werden im folgenden den Ausdruck „in bezug auf $\varkappa$ und $\varkappa'$" im allgemeinen weglassen (auch bei anderen, von $\varkappa$ und $\varkappa'$ abhängigen Begriffen). Die Menge der aufnehmbaren Bogensysteme von Γ bezeichnen wir mit M. Da $H=0$ zu M gehört, ist M nie leer. Aus (a) folgt, daß jedes aufnehmbare System, falls es mehrere Bogen enthält, aus lauter verschiedenen Bogen besteht.

Man kann behaupten: Es existiert der Wert

$$\nu_{\max} = \max_{H \in M} \nu(H)$$

$$(\nu_{\max} \text{ hängt von } \Gamma, \varkappa \text{ und } \varkappa' \text{ ab}),$$

der „die maximale Anzahl der aufnehmbaren Bogen" angibt.

(3.2) Wir wollen jedem Gewichtssystem $q = q(A, B, C)$ als seinen *Wert* bezüglich $\varkappa$ und $\varkappa'$ eine *ganze* Zahl $S(q)$ zuordnen. Kann ein Gewicht maximalerweise zu der Füllung von μ Bogen eines aufnehmbaren Bogensystems beitragen, so werden wir dies mit der Multiplizität μ in Betracht nehmen. Ist jedoch $H \in M$, so kann ein Punkt X auf höchstens $\varkappa(X) + \varkappa'(X)$ Bogen von H liegen und es können in X höchstens $\varkappa(X)$ Bogen von H enden, ferner kann eine Kante in höchstens einem Bogen von H liegen. Die angegebenen Grenzen können im allgemeinen nicht durch kleinere ersetzt werden. Deshalb bekommt das zu einem Punkt X gehörige Gewicht 1 bzw. 1/2 die Multiplizität $\varkappa(X) + \varkappa'(X)$ bzw. $\varkappa(X)$. (Nach (2.2) kann das zu X gehörige halbe Gewicht nur zur Füllung der in X endenden Bogen beitragen!) Ferner enthalten alle zu den Kanten gehörenden Gewichte die Multiplizität 1.

Die mit Multiplizitäten betrachteten Gewichte sollen jedoch nicht einfach addiert werden. Statt dessen teile man zuerst die halben Gewichte so in Klassen ein, daß je zwei „benachbarte" immer zur selben Klasse gehören; dann berechne man einzeln die *ganzen Teile* der Summe der zur selben Klasse gehörigen Gewichte, und endlich addiere man diese zu der Summe der ganzen Gewichte. Genauer gesagt, betrachten wir die Komponenten $[C_i]$ $(i = 1, \ldots, m)$ von $[C]$, und reihen für ein jedes i die zu den C_i-Punkten und zu den AC_i-Kanten gehörigen halben Gewichte in eine Klasse ein. Wir definieren also $S(q)$ folgendermaßen:[5]

$$S(q) = \varkappa(B) + \varkappa'(B) + \nu(A, A) + \sum_{i=1}^{m} \left[\frac{\varkappa(C_i) + \nu(A, C_i)}{2} \right]. \tag{1}$$

[5] $[a]$ bedeutet den ganzen Teil der Zahl a.

Wir wollen diesen Ausdruck noch auf eine andere Form bringen. Wir nennen eine C-Komponente $[C_i]$ *gerade* oder *ungerade*, je nachdem ob die Zahl $\varkappa(C_i)+\nu(A, C_i)$ gerade oder ungerade ist, und bezeichnen die Anzahl der ungeraden C-Komponenten mit τ_q. Es gilt dann

$$(2) \qquad S(q)=\varkappa(B)+\varkappa'(B)+\nu(A, A)+\frac{1}{2}(\varkappa(C)+\nu(A, C)-\tau_q).$$

Wir können behaupten: Es existiert der Wert

$$(3) \qquad S_{\min}=\min_{q\in Q} S(q)$$

($S_{\min}$ hängt von Γ, $\varkappa$ und $\varkappa'$ ab).

Nun sind wir endlich in der Lage, unseren Hauptsatz formulieren zu können:

HAUPTSATZ (3.3) *Es seien in den Punkten des nichtleeren Graphen Γ die Kapazitätsfunktionen $\varkappa$ und $\varkappa'$ definiert. Es gilt dann*

$$\nu_{\max}=S_{\min},$$

oder anders ausgedrückt: die maximale Anzahl der aufnehmbaren Bogen ist gleich dem Minimum der Werte der füllenden Gewichtssysteme.

Den Beweis des Hauptsatzes werden wir im II. Abschnitt durchführen.

BEMERKUNGEN. (1) Man kann unseren Hauptsatz in solcher Weise verallgemeinern, daß man auch den Kanten nichtnegative ganze Durchgangskapazitäten zuordnet. Bezeichnet $\varkappa'(x)$ die zur Kante x gehörige Kapazität, so muß man bei der Definition der aufnehmbaren Bogensysteme die Forderung (3.1) (a) durch jene ersetzen, daß jede Kante x zu höchstens $\varkappa'(x)$ Bogen des Systems gehören darf, und bei der Berechnung von $S(q)$ das zu x gehörige Gewicht mit der Multiplizität $\varkappa'(x)$ in Betracht nehmen. Man kann einen Beweis des so entstehenden Satzes dadurch erhalten, daß man die Kanten mit $\varkappa'(x)=0$ wegläßt, zu jeder Kante x mit $\varkappa'(x)>1$ genau $\varkappa'(x)-1$ neue Kanten mit denselben Randpunkten hinzunimmt und auf den so entstehenden Graphen den Satz (3.3) anwendet.

(2) Es entsteht ein interessantes Problem dadurch, daß man statt Bogensysteme nur Wegsysteme zuläßt. Wir vermuten, daß auch in diesem Falle ein ähnlicher Satz wie (3.3) besteht.

II. BEWEIS DES HAUPTSATZES

§ 4. Eine Umformung des Problems

Wir nehmen im ganzen II. Abschnitt an, daß Γ ein (nichtleerer) Graph ist, in dessen Punkten die Kapazitätsfunktionen $\varkappa$ und $\varkappa'$ definiert sind.

Es gilt folgende Behauptung:

(4. 1) *Ist $H \in M$ und $q \in Q$, so ist $\nu(H) \leqq S(q)$.*

Beweis. Es sei $H \in M$ und $q = q(A, B, C)$, und es seien $[C_i]$ $(i = 1, \ldots, m)$ die C-Komponenten. Wir zerlegen H in drei Bogensysteme H_a, H_b und H_c. H_b bestehe aus sämtlichen solchen Bogen von H, die mindestens einen B-Punkt enthalten; H_a aus denen, die keinen B-Punkt, aber mindestens eine AA-Kante enthalten; endlich H_c aus denen, die weder B-Punkte, noch AA-Kanten enthalten. H_a, H_b und H_c können auch leer sein. Es gelten dann

$$\nu(H_a) + \nu(H_b) + \nu(H_c) = \nu(H),$$

$$\nu_1'(H_a) \leqq \nu(A, A) \quad \text{und} \quad \nu(H_b) \leqq \varkappa(B) + \varkappa'(B).$$

Es sei H_i $(i = 1, \ldots, m)$ das System sämtlicher Bogen von H_c, die C_i-Punkte enthalten. Im Falle $C = \emptyset$ sei $H_1 = 0$. Es gilt

$$\sum_{i=1}^{m} \nu(H_i) \geqq \nu(H_c). \tag{2}$$

Ferner bezeichne λ_i, μ_i bzw. ν_i die Anzahl derjenigen Bogen von H_i, die genau 2, 1 bzw. 0 Randpunkte in C_i haben. Dann gelten

$$\lambda_i + \mu_i + \nu_i = \nu(H_i), \tag{3}$$

$$2\lambda_i + \mu_i \leqq \varkappa(C_i) \tag{4}$$

$(i = 1, \ldots, m)$.

Fällt nur ein bzw. kein Randpunkt des Bogens h von H_i in C_i, so enthält h mindestens eine bzw. zwei AC_i-Kanten. Es gilt daher $\mu_i + 2\nu_i \leqq \nu(A, C_i)$, woraus nach (4)

$$\lambda_i + \mu_i + \nu_i \leqq \left[\frac{\varkappa(C_i) + \nu(A, C_i)}{2}\right] \qquad (i = 1, \ldots, m) \tag{5}$$

folgt. Aus (1), (2), (3) und (5) ergibt sich $\nu(H) \leqq S(q)$.

(4. 2) Es sei $H \in M$. Wir setzen

$$\delta_X(H) = \varkappa(X) - [H, X], \quad \delta(H) = \sum_{X \in \Phi} \delta_X(H), \quad \delta_{\min} = \min_{H \in M} \delta(H).$$

Der Wert $\delta_{\min}$ existiert und es gilt $\delta_X(H) \geqq 0$, $\delta(H) \geqq 0$, $\delta_{\min} \geqq 0$. Nach (2. 1) (1) gelten

(1) $$\delta(H) = \varkappa(\Phi) - 2\nu(H), \quad \delta_{\min} = \varkappa(\Phi) - 2\nu_{\max}.$$

Es ist ferner

(2) $$\nu_{\max} \leqq \frac{1}{2}\varkappa(\Phi).$$

Ist $q \in Q$, so folgt nach (4. 1) aus (1)

(3) $$S(q) \geqq \frac{1}{2}(\varkappa(\Phi) - \delta_{\min}).$$

Wir werden nun unseren Hauptsatz (3. 3) dadurch beweisen, daß wir ein solches q von Q konstruieren, welches die Gleichung

(4) $$S(q) = \frac{1}{2}(\varkappa(\Phi) - \delta_{\min})$$

befriedigt. Es folgt nämlich dann aus (3), das $S_{\min} = 1/2(\varkappa(\Phi) - \delta_{\min})$, was nach (1) mit $\nu_{\max} = S_{\min}$ gleichbedeutend ist.

§ 5. Eine weitere Umformung. Ketten

Um ein q zu bestimmen, welches (4. 2) (4) befriedigt, wollen wir eine weitere Umformung durchführen, die kurzgesagt darin besteht, daß man die Bogensysteme durch gewisse, zu diesen Systemen gehörige Teilgraphen von Γ ersetzt.

(5. 1) Wir wollen in dieser Arbeit unter einer *Kette* von Γ einen solchen Teilgraphen von Γ verstehen, der jeden Punkt von Γ enthält. Eine Kette von Γ ist durch die Angabe seiner Kanten eindeutig bestimmt. Deshalb sagen wir auch, daß eine Kette aus seinen Kanten „besteht". Diejenige Kette, die keine Kante enthält, heißt die *Nullkette* von Γ. Im folgenden sollen die vorkommenden Ketten, wenn anders nicht gesagt wird, immer zu Γ gehörige Ketten bedeuten.

Mit f (auch mit Indizes oder anderen zeichen versehen) bezeichnen wir immer Ketten. Ist $f' \subseteq f$, so soll $f - f'$ diejenige Kette bezeichnen, die aus sämtlichen solchen Kanten von f besteht, welche in f' nicht vorkommen.

Das Zeichen (f, X) soll die Anzahl derjenigen Kanten von f bedeuten, die zu X inzident sind (d. h. es ist $(f, X) = \varrho_f(X)$).

Ist h ein Bogen, H ein Bogensystem von Γ, so soll $\tilde{h}$ bzw. $\tilde{H}$ diejenige Kette bedeuten, die aus den Kanten von h bzw. aus den Kanten sämtlicher

Bogen von H besteht. (Ist $H=0$, so ist $\tilde{H}$ die Nullkette.) Nach (2. 1) und (3. 1) (a) gilt dann für jedes H von M und für jedes X

(1) $$(\tilde{H}, X) = [H, X] + 2|H, X|.$$

(5. 2) Ist $H \in M$ und gibt es ein H_1 in M mit $\tilde{H}_1 \subset \tilde{H}$ und $\delta(H_1) = \delta(H)$, so heißt H *reduzibel*. Ist $H \in M$ und ist es nicht reduzibel, so heißt es *irreduzibel*. Es bezeichne M_i die Menge der irreduziblen Systeme von M. Das System $H=0$ ist ein Element von M_i und wir können behaupten, daß zu jedem H von M ein H_1 in M_i mit $\tilde{H}_1 \subseteq \tilde{H}$ und $\delta(H_1) = \delta(H)$ existiert. Daraus folgt

(1) $$\min_{H \in M_i} \delta(H) = \delta_{\min}.$$

Es gilt ferner

(5. 3) *Ist $H \in M_i$ und besteht für den Punkt X die Ungleichung $|H, X| > 0$, so ist $[H, X] = \varkappa(X)$.*

Beweis. Gilt nämlich für X $|H, X| > 0$, so gibt es einen Bogen h von H, der X als inneren Punkt enthält. h enthält einen solchen Teilbogen h', daß der eine Randpunkt von h' der Punkt X ist, der andere jedoch mit einem Randpunkt von h zusammenfällt. Ersetzt man dann in H den Bogen h durch h', so entsteht ein solches System H_1, das im Falle $[H, X] < \varkappa(X)$ den Bedingungen $H_1 \in M$, $\tilde{H}_1 \subset \tilde{H}$ und $\nu(H_1) = \nu(H)$ genügt. Dies widerspricht jedoch nach (4. 2) (1) der Irreduzibilität von H.

Aus (5. 1) (1) und (5. 3) folgt

(5. 4) *Es sei $H \in M_i$. Dann gelten*

(1) *im Falle $(\tilde{H}, X) \geqq \varkappa(X)$ die Gleichungen $[H, X] = \varkappa(X)$ und $(\tilde{H}, X) - \varkappa(X) = 2|H, X|$;*

(2) *im Falle $(\tilde{H}, X) \leqq \varkappa(X)$ die Gleichung $|H, X| = 0$.*

Aus (5. 1) (1) und (5. 4) folgt weiter

(5. 5) *Ist $H \in M_i$, so besitzt $\tilde{H}$ die folgenden Eigenschaften: Für jedes X ist $(\tilde{H}, X) \leqq \varkappa(X) + 2\varkappa'(X)$. Ist $(\tilde{H}, X) > \varkappa(X)$, so ist die Zahl $(\tilde{H}, X) - \varkappa(X)$ gerade.*

(5. 6) Wir wollen unter Beachtung von (5. 5) eine beliebige Kette f (von Γ in bezug auf $\varkappa$ und $\varkappa'$) *aufnehmbar* nennen, wenn f den folgenden Bedingungen genügt:

(a) Für jedes X gilt $(f, X) \leqq \varkappa(X) + 2\varkappa'(X)$.

(b) Ist $(f, X) > \varkappa(X)$, so ist die Zahl $(f, X) - \varkappa(X)$ gerade.

Die Menge der aufnehmbaren Ketten (von Γ) bezeichnen wir mit $\tilde{M}$. Die Nullkette ist ein Element von $\tilde{M}$.

Es sei $f \in \tilde{M}$. Gilt dann für den Punkt X $(f, X) \geqq \varkappa(X)$ bzw. $(f, X) > \varkappa(X)$, so sagen wir, daß X durch f *gefüllt* bzw. *übergefüllt* ist.

Wir setzen

$$\delta_X(f) = \max(0, \varkappa(X) - (f, X)), \quad \delta(f) = \sum_{X \in \Phi} \delta_X(f), \quad \tilde{\delta}_{\min} = \min_{f \in \tilde{M}} \delta(f)$$

und nennen eine Kette von $\tilde{M}$ mit $\delta(f) = \tilde{\delta}_{\min}$ eine *extreme* Kette von $\tilde{M}$. Offensichtlich existieren extreme Ketten in $\tilde{M}$.

Man kann behaupten:

(5.7) *Ist $f' \subseteq f$, so gilt $\delta(f') \geqq \delta(f)$, und das Gleichheitszeichen gilt dann und nur dann, wenn für jedes X mit $(f', X) < (f, X)$ die Ungleichung $(f', X) \geqq \varkappa(X)$ besteht.*

Aus (5.4), (5.5) und (5.6) folgt

(5.8) *Ist $H \in M_i$, so ist $\tilde{H} \in \tilde{M}$ und es gilt für jedes X $\delta_X(\tilde{H}) = \delta_X(H)$, sowie $\delta(\tilde{H}) = \delta(H)$.*

(5.9) Ist $f \in \tilde{M}$ und gibt es ein f' in $\tilde{M}$ mit $f' \subset f$ und $\delta(f') = \delta(f)$, so heißt f *reduzibel*. Ist $f \in \tilde{M}$ und nicht reduzibel, so heißt es *irreduzibel*. Die Menge der irreduziblen Ketten von $\tilde{M}$ bezeichnen wir mit $\tilde{M}_i$. Die Nullkette ist ein Element von $\tilde{M}_i$, und zu jedem f von $\tilde{M}$ gibt es ein f' in $\tilde{M}_i$ mit $f' \subseteq f$ und $\delta(f') = \delta(f)$. Daraus folgt

$$\min_{f \in \tilde{M}_i} \delta(f) = \tilde{\delta}_{\min}. \tag{1}$$

Ist $f \in \tilde{M}$ und ist k ein solcher Kreis von f, dessen sämtliche Punkte durch f übergefüllt sind, so heißt k ein *weglaßbarer* Kreis von f. Nach (5. 7) können wir behaupten:

(5.10) *Ist $f \in \tilde{M}$ und gibt es einen weglaßbaren Kreis von f, so ist f reduzibel.*

Es gilt auch die Umkehrung dieser Behauptung:

(5.11) *Ist f eine reduzible Kette von $\tilde{M}$, so enthält f einen weglaßbaren Kreis von f.*

Beweis. Existiert nämlicht ein f' in $\tilde{M}$ mit $f' \subset f$ und $\delta(f') = \delta(f)$, so sind nicht alle Punkte in $f'' = f - f'$ isoliert, und sämtliche nichtisolierten Punkte von f'' sind nach (5.7) durch f' gefüllt und durch f übergefüllt. Ferner sind alle diese Punkte nach (5.6) (b) geraden Grades in f''. Daraus folgt, daß f'' einen Kreis enthält. Dieser ist aber von f weglaßbar.

Ähnlicherweise kann man die Richtigkeit folgender Behauptung einsehen:

(5.12) *Ist $f \in \tilde{M}$ und ist f nicht die Nullkette, sind ferner sämtliche nicht-isolierten Punkte von f durch f übergefüllt, so enthält f einen weglaßbaren Kreis.*

Wir beweisen nun folgende Behauptung:

(5.13) *Zu jedem f von $\tilde{M}_i$ gibt es ein H in M_i mit $\tilde{H}=f$.*

Beweis. (I) Es sei $f \in \tilde{M}_i$. Ist f die Nullkette, so können wir $H=0$ setzen. Daher kann angenommen werden, daß f auch nichtisolierte Punkte enthält. Wir wollen in diesem Beweis einen solchen Bogen von f, dessen innere Punkte durch f übergefüllt sind, dessen Randpunkte jedoch nicht, einen *f-Bogen* nennen. Wir zeigen, daß f-Bogen existieren. Nach (5.12) und (5.10) gibt es ein X_0 mit $0<(f, X_0) \leqq \varkappa(X_0)$. f enthält eine Kante $[X_0 x_1 X_1]$. Ist $(f, X_1) \leqq \varkappa(X_1)$, so ist $[X_0 x_1 X_1]$ ein f-Bogen. Nehmen wir an, daß $(f, X_1) > \varkappa(X_1)$ ist. Es bezeichne L die Menge derjenigen Wege w von f, die folgende Eigenschaften besitzen: X_0 ist der eine Randpunkt von w, und sämtliche von X_0 verschiedenen Punkte von w sind durch f übergefüllt. L ist nichtleer. Es sei $w'=[X_0 x_1 X_1 \ldots X_{n-1} x_n X_n]$ ein Weg von L mit maximaler Anzahl von Kanten. Es gibt eine von x_n verschiedene Kante $[X_n x' X']$ von f. Es ist dann $h=[X_0 x_1 X_1 \ldots X_{n-1} x_n X_n x' X']$ ein f-Bogen. Ist nämlich h ein Weg, so muß $(f, X') \leqq \varkappa(X')$ sein. Im Falle $X'=X_0$ ist unsere Behauptung trivial. Der Fall $X'=X_i$ $(0<i \leqq n-1)$ kann nicht eintreten, da dann $[X_i x_{i+1} X_{i+1} \ldots \ldots X_n x' X']$ ein weglaßbarer Kreis von f wäre.

(II) Es bezeichne m eine positive ganze Zahl. Wir nehmen an, daß unser Satz für sämtliche solche Ketten von $\tilde{M}_i$ richtig ist, die weniger als m Kanten enthalten, und daß f $(f \in \tilde{M}_i)$ genau m Kanten besitzt. Es sei h_1 ein beliebiger f-Bogen. Für $f_1=f-\tilde{h}_1$ gilt $f_1 \in \tilde{M}$ und $\delta(f_1)=\delta(f)+2$. f_1 kann keinen weglaßbaren Kreis enthalten, denn ein solcher Kreis wäre auch von f weglaßbar. Nach (5.11) ist dann $f_1 \in \tilde{M}_i$. f_1 enthält weniger Kanten als f, und so gibt es laut unserer Annahme ein H_1 in M_i mit $\tilde{H}_1=f_1$. Nach (5.8) ist dann für jedes X $\delta_X(H_1)=\delta_X(f_1)$ und $\delta(H_1)=\delta(f_1)$. Fügen wir h_1 zu H_1, so entsteht ein System H mit $\tilde{H}=f$.

Wir zeigen, daß $H \in M$ ist. H genügt der Bedingung (3.1) (a). Um zu beweisen, daß H auch (3.1) (b) erfüllt, genügt es nur die Punkte von h_1 zu untersuchen. Ist X ein innerer Punkt von h_1, so ist $(f_1, X) \geqq \varkappa(X)$, daher gilt nach (5.4) $(f_1, X)=\varkappa(X)+2|H_1, X|$, was zusammen mit $(f_1, X)=(f, X)-2< <\varkappa(X)+2\varkappa'(X)$ die Ungleichung $|H_1, X|<\varkappa'(X)$ ergibt. Ist X ein Randpunkt von h_1, so ist $(f, X) \leqq \varkappa(X)$, und daher gilt $[H, X] \leqq (\tilde{H}, X) \leqq \varkappa(X)$. Daraus folgt, daß H die Bedingung (3.1) (b) erfüllt. Es gilt also $H \in M$.

Es gilt ferner $\delta(H)=\delta(H_1)-2=\delta(f)$.

Endlich beweisen wir, daß H irreduzibel ist. Nehmen wir nämlich das Gegenteil an, dann gibt es ein H' in M_i mit $\tilde{H}' \subset \tilde{H}$ und $\delta(H')=\delta(H)$. Nach (5.8) ist $\tilde{H}' \in \tilde{M}$ und $\delta(\tilde{H}')=\delta(H')$, woraus $\delta(\tilde{H}')=\delta(f)$ folgt. Das widerspricht jedoch der Irreduzibilität von f. Es ist damit der Beweis von (5.13) beendet.

(5. 14) *Ist* $H \in M_i$, *so ist* $\tilde{H} \in \tilde{M}_i$.

Beweis. Laut (5. 8) genügt es zu zeigen, daß $\tilde{H}$ irreduzibel ist. Nehmen wir an, daß es ein f' mit $f' \in \tilde{M}$, $f' \subset \tilde{H}$ und $\delta(f') = \delta(\tilde{H})$ gibt. Zu f' existiert ein f'' in $\tilde{M}_i$ mit $f'' \subseteq f'$ und $\delta(f'') = \delta(f')$. Nach (5. 13) gibt es dann ein H' in M_i mit $\tilde{H}' = f''$. Laut (5. 8) ist $\delta(H') = \delta(f'')$ und $\delta(\tilde{H}) = \delta(H)$. Es gilt also $\tilde{H}' \subset \tilde{H}$ und $\delta(H') = \delta(H)$. Das widerspricht aber der Irreduzibilität von H.

(5. 15) Aus (5. 8), (5. 13) und (5. 14) folgt

$$\min_{f \in \tilde{M}_i} \delta(f) = \min_{H \in M_i} \delta(H),$$

und das ergibt, zusammen mit (5. 2) (1) und (5. 9) (1) die Gleichung

$$\tilde{\delta}_{\min} = \delta_{\min}.$$

Um unseren Hauptsatz zu beweisen, genügt es daher nach (4. 2) ein solches Gewichtssystem q von Q zu konstruieren, welches die Gleichung

$$S(q) = \frac{1}{2}(\varkappa(\Phi) - \tilde{\delta}_{\min}) \tag{1}$$

befriedigt. Zu dieser Konstruktion werden wir von einer extremen Kette f von $\tilde{M}$ ausgehen, und auf diese das Verfahren der alternierenden Züge anwenden.

§ 6. Alternierende Züge

(6. 1) Um unsere Ausdrucksweise zu verkürzen, ist es vorteilhaft, einer jeden Kette f von Γ eine *charakteristische Funktion* $f(x)$ zuzuordnen. $f(x)$ sei auf sämtlichen Kanten von Γ definiert, und es bestehe $f(x) = 1$ oder $f(x) = 0$, je nachdem ob x eine Kante von f ist oder nicht.

Es sei f vorläufig ein beliebiges Element von $\tilde{M}$. Wir nennen den Zug $p = (X_0 x_1 X_1 \ldots X_{n-1} x_n X_n)$ einen *zu f gehörigen alternierenden Zug* (kurz: a-Zug), wenn entweder $n = 1$ ist, oder wenn $n > 1$ ist und p für jedes i $(1 \leqq i \leqq n-1)$ folgenden Bedingungen genügt:

(a) Ist $f(x_i) = f(x_{i+1})$, so ist X_i ein einfacher Punkt von p.
(b) Ist $f(x_i) = f(x_{i+1}) = 1$, so ist $(f, X) > \varkappa(X)$.
(c) Ist $f(x_i) = f(x_{i+1}) = 0$, so ist $(f, X) \leqq \varkappa(X) + 2\varkappa'(X) - 2$.

Wir nennen den Punkt X des a-Zuges p einen *Wechselpunkt* von p, wenn bei dem Durchlaufen von p der Wert von $f(x)$ sich bei *jedem* Durchgang über X ändert, oder wenn X ein Randpunkt von p ist. Nach (a) ist jeder mehrfache Punkt von p ein Wechselpunkt von p.

Ist $p=(X_0x_1X_1\ldots X_{n-1}x_nX_n)$ ein zu f gehöriger a-Zug, so ist auch $\bar{p}=(X_nx_nX_{n-1}\ldots X_1x_1X_0)$, sowie

$$p_{ij}=(X_{i-1}x_iX_i\cdots X_{j-1}x_jX_j) \qquad (1\leqq i\leqq j\leqq n)$$

ein solcher Zug. Die Züge p_{1i} bzw. p_{in} $(1\leqq i\leqq n)$ heißen die *Anfangs-* bzw. *Endteile* von p.

Man kann in gewissen Fällen zwei zu f gehörige a-Züge zu einem dritten vereinigen. Wir wollen einige leicht ersichtliche hinreichende Bedingungen solcher Vereinigungen in dem folgenden Lemma zusammenfassen:

LEMMA (6. 2) *Es seien* $p=(X_0x_1X_1\ldots X_{m-1}x_mX_m)$ *und* $p'=(X_0'x_1'X_1'\ldots$ $\ldots X_{n-1}'x_n'X_n')$ *zwei zu* f *gehörige a-Züge, die den folgenden vier Bedingungen genügen:* (1) $X_m=X_0'$; (2) p *und* p' *haben keine gemeinsamen Kanten;* (3) *ist* X *ein gemeinsamer Punkt von* p *und* p', *so ist* X *sowohl in* p *wie auch in* p' *ein Wechselpunkt;* (4) *Es besteht einer der folgenden drei Fälle:*

(a) $f(x_m)\neq f(x_1')$;

(b) $f(x_m)=f(x_1')=1$, X_m *ist sowohl in* p *als auch in* p' *einfach und* $(f, X_m)>\varkappa(X_m)$;

(c) $f(x_m)=f(x_1')=0$, X_m *ist in* p *und auch in* p' *einfach und* $(f, X_m)\leqq$ $\leqq\varkappa(X_m)+2\varkappa'(X_m)-2$.

Dann ist auch

$$pp'=(X_0x_1X_1\ldots X_{m-1}x_mX_mx_1'X_1'\ldots X_{n-1}'x_n'X_n')$$

ein zu f *gehöriger a-Zug.*

Der folgende Satz enthält die Grundidee der Anwendung der a-Züge.

SATZ (6. 3) *Ist* f *eine extreme Kette von* $\tilde{M}$, *so gibt es keinen zu* f *gehörigen a-Zug* $p=(X_0x_1X_1\ldots X_{n-1}x_nX_n)$ *mit den Eigenschaften:* $f(x_1)=$ $=f(x_n)=0$, $(f, X_0)<\varkappa(X_0)$, $(f, X_n)<\varkappa(X_n)$ *und im Falle* $X_0=X_n$ $(f, X_0)\leqq$ $\leqq\varkappa(X_0)-2$.

BEWEIS. Es sei $f\in\tilde{M}$ und $\delta(f)=\tilde{\delta}_{\min}$. Nehmen wir an, daß solche zu f gehörigen a-Züge existieren, welche die in (6. 3) angeführten Eigenschaften besitzen, und es sei $p=(X_0x_1X_1\ldots X_{n-1}x_nX_n)$ ein solcher Zug mit minimaler Anzahl von Kanten. Wir definieren eine Kette f' wie folgt: Ist $f(x_i)=1$ $(1\leqq i\leqq n)$, so sei $f'(x_i)=0$. Ist $f(x_i)=0$ $(1\leqq i\leqq n)$, so sei $f'(x_i)=1$. Ist $x\neq x_1,\ldots,x_n$, so sei $f'(x)=f(x)$. Wir werden zeigen, daß $f'\in\tilde{M}$ und $\delta(f')<\delta(f)$ ist, woraus die Richtigkeit unseres Satzes folgt.

(I) Zuerst sei X entweder ein Wechselpunkt von p, oder sei X kein Punkt von p. Fällt X mit einem der Punkte X_0 und X_n zusammen, so gilt $(f, X)<(f', X)\leqq\varkappa(X)$, und daraus folgt $\delta_X(f')<\delta_X(f)$. Ist $X\neq X_0, X_n$, so

ist $(f', X) = (f, X)$. Es ist dann $\delta_X(f') = \delta_X(f)$, und wir sehen, daß f' bei X beide Bedingungen von (5.6) erfüllt.

(II) Es sei $X = X_i$ $(0 < i < n)$, und es gelte $f(x_{i-1}) = f(x_i) = 1$. Dann folgt nach (6.1)(a), (b): $X \neq X_0, X_n$, $(f', X) = (f, X) - 2$ und $(f, X) > \varkappa(X)$. Daraus ergibt sich $\delta_X(f') = \delta_X(f) = 0$ und man sieht, daß f' bei X beide Bedingungen unter (5.6) befriedigt.

(III) Es sei $X = X_i$ $(0 < i < n)$ und $f(x_{i-1}) = f(x_i) = 0$. Dann folgt nach (6.1)(a), (c): $X \neq X_0, X_n$, $(f', X) = (f, X) + 2$ und $(f, X) \leqq \varkappa(X) + 2\varkappa'(X) - 2$. Daraus kann man sehen, daß $\delta_X(f') \leqq \delta_X(f)$ gilt, sowie daß f' bei X der Bedingung (a) unter (5.6) genügt. Gilt für X noch $(f, X) \geqq \varkappa(X)$, so folgt aus den obigen, daß f' bei X auch (5.6)(b) erfüllt. Der andere Fall kann nicht eintreten. Wäre nämlich $(f, X) < \varkappa(X)$, so wäre — da X_i ein einfacher Punkt von p ist — $(X_0 x_1 X_1 \dots X_{i-1} x_{i-1} X_i)$ ein zu f gehöriger a-Zug, mit den in (6.3) angeführten Eigenschaften, der weniger Kanten als p enthalten würde.

Aus (I), (II) und (III) folgt $f' \in \tilde{M}$ und $\delta(f') < \delta(f)$.

§ 7. Der Hilfspunkt X^*. Beweis eines Lemmas

(7.1) Es sei nun f eine beliebige *extreme* Kette von $\tilde{M}$. Im weiteren Teil des II. Abschnittes wollen wir diese Kette f festhalten. Um unsere Beweisführung zu vereinfachen, konstruieren wir nach BERGE ([3], S. 176) aus Γ, $\varkappa$ und f einen neuen Graphen Γ^* wie folgt: Wir nehmen zu Γ einen neuen Punkt X^* hinzu und verbinden X^* mit jedem durch f *nicht gefüllten* Punkt X von Γ durch $\delta_X(f)$ neue Kanten. Ferner erweitern wir die Funktionen $\varkappa$ und $\varkappa'$ auf Γ^* folgendermaßen: Es sei

(a) $\varkappa(X^*) = \tilde{\delta}_{\min}$; (b) $\varkappa'(X^*) = 0$.

Die zu Γ^* gehörige Kette f^* soll aus sämtlichen Kanten von f und aus sämtlichen neuen Kanten bestehen. (f^*, X) soll die Anzahl der zu X inzidenten Kanten von Γ^* bezeichnen. Wir stellen einige leicht ersichtliche Eigenschaften von f^* zusammen:

(1) Für jede zu X^* inzidente Kante x gilt $f^*(x) = 1$.

(2) $(f^*, X^*) = \varkappa(X^*)$.

(3) f^* ist eine (bezüglich $\varkappa$ und $\varkappa'$) aufnehmbare Kette von Γ^*.

(4) Jeder Punkt X von Γ^* ist durch f^* gefüllt.

(5) Ist X durch f^* übergefüllt, so gibt es keine XX^*-Kante.

Aus (6.3) leiten wir eine weitere Eigenschaft von f^* ab:

(7.2) *Es existiert in Γ^* kein zu f^* gehöriger a-Zug p^* von der Struktur $p^* = (X^* x_0 X_0 \dots X_n x_{n+1} X^*)$ $(n \geqq 0)$.*

Beweis. Nehmen wir an, daß in Γ^* ein a-Zug p^* von der angegebenen Struktur existiert. Ist $n=0$, so ist $p^*=(X^*x_0X_0x_1X^*)$. Nach (7. 1)(1) und (6. 1)(b) gilt $(f^*, X_0)>\varkappa(X_0)$, was zu (7. 1)(5) in Widerspruch steht.

Es sei $n>0$. Dann ist nach (7. 1)(1), (2) und (6. 1)(b) der Teil $p=(X_0x_1X_1\ldots X_{n-1}x_nX_n)$ von p^* ein zu f gehöriger a-Zug von Γ. Aus der Existenz von x_0 und x_{n+1} folgt, daß X_0 und X_n durch f nicht gefüllt sind, und im Falle $X_0=X_n$ sogar $(f, X_0)\leqq\varkappa(X_0)-2$ besteht. Ferner gilt nach (7.1)(4), (5) $(f^*, X_0)=\varkappa(X_0)$, $(f^*, X_n)=\varkappa(X_n)$, und so erhalten wir unter Beachtung von (7. 1)(1) und (6. 1)(b) die Gleichung $f(x_1)=f(x_n)=0$. Dies widerspricht jedoch dem Satz (6. 3).

(7. 3) Von hier an bis zum Ende des II. Abschnittes sollen sich unsere Begriffe und Bezeichnungen — wenn anderes nicht gesagt wird — nicht auf Γ, sondern auf Γ^* beziehen. Unter Punkten, Kanten und Zügen verstehen wir also immer beliebige Punkte, Kanten und Züge von Γ^*. Ferner haben $\nu(E, F)$ und $[E]$ die Bedeutung $\nu_{\Gamma^*}(E, F)$ bzw. $[E]_{\Gamma^*}$, und eine E-Komponente bedeutet eine E-Komponente von Γ^*.

Um unsere Ausdrucksweise weiter zu vereinfachen, wollen wir die Kanten von f^* *α-Kanten*, diejenigen Kanten, die nicht zu f^* gehören, *β-Kanten* nennen. (Für jede α-Kante ist also $f^*(x)=1$, und für jede β-Kante $f^*(x)=0$.) Die Zeichen $u, v, w, \bar{u}, \bar{v}, \bar{w}$ sollen einen beliebigen der Buchstaben α und β bedeuten, und zwar soll im selben Satz bzw. Beweis immer $\bar{u}\neq u$, $\bar{v}\neq v$, $\bar{w}\neq w$ gelten.

Sämtliche im weiteren Teil des II. Abschnittes vorkommenden a-Züge sind zu f^* gehörige a-Züge von Γ^*. Deshalb werden wir diese kurz nur als a-Züge bezeichnen. Vielmals wird es genügen, statt der Randkanten der a-Züge nur die Art dieser Kanten zu bezeichnen. Wir führen daher folgende kurze Schreibweise ein:

$$p=(X_1^u\ldots{}^vX_2)$$

soll bedeuten, daß X_1 der Anfangspunkt, X_2 der Endpunkt des a-Zuges p ist und die Anfangs- bzw. Endkante von p eine u- bzw. v-Kante ist. Kann kein Mißverständnis entstehen, so werden wir im allgemeinen in den a-Zügen neben den Randpunkten nur die in unseren Behauptungen explizit vorkommenden Elemente bzw. Kantenarten bezeichnen. Wir wiederholen nochmals die folgende Vereinbarung von (6. 1): p und $\bar{p}$ bezeichnen solche a-Züge, die sich nur in der Durchlaufsrichtung unterscheiden.

Im Beweis des folgenden Lemmas benützen wir keine speziellen Eigenschaften von Γ^*, $\varkappa$ und $\varkappa'$. Bezüglich f^* benützen wir nur (7. 1)(3).

Lemma (7. 4) *Es sollen die a-Züge $p=(X_1^u\ldots{}^vX_2)$ und $p_0=(X_0x_0\ldots X')$ die folgenden Bedingungen erfüllen:*

(1) *X' liegt auf p;*

(2) *liegt X_0 auf p, so ist er ein Wechselpunkt von p.*

Dann existiert ein a-Zug von der Struktur

$$p_1 = (X_0 x_0 \dots {}^u X_1) \quad \textit{oder} \quad p_2 = (X_0 x_0 \dots {}^v X_2),$$

der nur solche Kanten enthält, die in p oder in p_0 liegen.

Beweis. (I) Liegt x_0 in p, so bezeichne p' bzw. p'' denjenigen Anfangs- bzw. Endteil von p, dessen End- bzw. Anfangskante x_0 ist. X_0 ist entweder der Endpunkt von p' oder der Anfangspunkt von p''. Im ersten Falle kann man $p_1 = \bar{p}'$, im zweiten $p_2 = p''$ setzen.

(II) Wir nehmen an, daß x_0 nicht in p liegt. Es sei $p_3 = (X_0 x_0 \dots x_3 X_3)$ der kürzeste Anfangsteil von p_0, dessen Endpunkt auf p liegt und es sei x_3 eine w-Kante. Da x_0 nicht auf p liegt, haben p_3 und p keine gemeinsame Kante. p_3 und p können nur die Punkte X_0 und X_3 gemeinsam haben, und im Falle $X_3 \neq X_0$ ist X_3 einfach in p_3.

(1) Ist $X_3 = X_1$ und $w = u$, so ist p_3 ein gesuchter Zug p_1. Ist $X_3 = X_1$ und $w \neq u$, so können wir nach (6.2)(a) $p_2 = p_3 p$ setzen. Im Falle $X_3 = X_2$ kann man ähnlich verfahren.

(2) Es sei $X_3 \neq X_1, X_2$. Es bezeichne p_4 einen Anfangsteil von p, dessen Endpunkt X_3 ist, und p_5 den zu p_4 komplementären Endteil von p. Es sei x_4 die Endkante von p_4, x_5 die Anfangskante von p_5. Wir können behaupten: Die gemeinsamen Punkte von p_3 und p_4 bzw. von p_3 und p_5 sind in den beiden Zügen Wechselpunkte.

Ist ferner X_3 ein Wechselpunkt von p, so besteht entweder $f^*(x_3) \neq$ $\neq f^*(x_4)$ oder $f^*(x_3) \neq f^*(x_5)$. Nach (6.2)(a) kann man im ersten Falle $p_1 = p_3 \bar{p}_4$, im zweiten $p_2 = p_3 p_5$ setzen.

Ist X_3 kein Wechselpunkt von p, so ist X_3 einfach in jedem der Züge p, p_4 und p_5. Ferner ist $X_3 \neq X_0$, und so ist X_3 auch in p_3 einfach. Man kann nun $p_2 = p_3 p_5$ (und auch $p_1 = p_3 p_4$) setzen. Dies folgt im Falle $f^*(x_3) \neq$ $\neq f^*(x_5)$ wieder aus (6.2)(a). Im Falle $f^*(x_3) = f^*(x_5)$ haben $f^*(x_3), f^*(x_4)$ und $f^*(x_5)$ den gleichen Wert 1 bzw. 0, und so folgt laut (6.1)(b), (c) aus (6.2)(b), (c) unsere Behauptung.

§ 8. Die α-, β- und γ-Punkte

(8.1) In diesem Paragraphen nützen wir von Γ^*, $\varkappa$ und $\varkappa'$ nur die eine spezielle Eigenschaft aus, daß $\varkappa'(X^*) = 0$ ist, und von f^* nur die Eigenschaften (7.1)(1), (2), (3) und (7.2).

Wir führen einige neue Bezeichnungen ein. Ein a-Zug, dessen Anfangspunkt X^* ist, soll ein *X^*-Zug* heißen. Existiert zu dem Punkt X ein X^*-Zug,

dessen Endpunkt X und dessen Endkante eine u-Kante ist, so heißt X *erreichbar*, genauer *u-erreichbar*. Unabhängig von dieser Definition wollen wir X^* immer erreichbar, und zwar β-erreichbar nennen. Nach (7.2) ist X^* nicht α-erreichbar. Es ist klar, daß jeder Punkt eines X^*-Zuges erreichbar ist.

(8.2) X soll ein *α-Punkt* heißen, wenn X α-erreichbar, jedoch nicht β-erreichbar ist und $(f^*, X) \leq \varkappa(X)$ besteht. Ist X β-erreichbar, jedoch nicht α-erreichbar, und gilt $(f^*, X) > \varkappa(X) + 2\varkappa'(X) - 2$, so wollen wir X einen *β-Punkt* nennen. Nach unseren Annahmen gilt:

(8.3) *X^* ist ein β-Punkt.*

Wir nennen einen jeden solchen Punkt von Γ^*, der weder α- noch β-Punkt ist, einen *γ-Punkt*. Die unerreichbaren Punkte sind alle γ-Punkte. Ferner sind sämtliche Punkte, die sowohl α- als auch β-erreichbar sind, ebenfalls γ-Punkte. Ein erreichbarer γ-Punkt kann jedoch auch nur α- oder nur β-erreichbar sein. Diese wollen wir γ_α- bzw. *γ_β-Punkte* nennen.

Wir können behaupten:

(8.4) *Ist X ein γ_α-Punkt, so gilt $(f^*, X) > \varkappa(X)$; ist X ein γ_β-Punkt, so gilt $(f^*, X) \leqq \varkappa(X) + 2\varkappa'(X) - 2$.*

Es besteht ferner:

(8.5) *Liegt X auf dem X^*-Zuge p, und ist X ein u-Punkt, so ist X ein Wechselpunkt von p.*

Beweis. Ist X kein Wechselpunkt von p, so ist X kein Randpunkt von p, und er ist einfach in p. Es gibt daher einen und nur einen Anfangsteil p' von p mit dem Endpunkt X. Da auch p' ein X^*-Zug ist, muß nach (8.2) die Endkante von p' eine u-Kante sein. Laut (6.1)(b), (c) widerspricht das jedoch der Annahme, daß X ein u-Punkt ist.

Aus (7.1), (1), (2) und (6.1)(b) folgt:

(8.6) *X^* kann nur ein Randpunkt eines a-Zuges sein.*

Es gilt ferner:

(8.7) *Ist X_1 ein $\bar{u}$-Punkt und X_2 ein $\bar{v}$-Punkt, so gibt es keinen a-Zug p von der Struktur $p = (X_1^u \ldots {}^v X_2)$.*

Beweis. Nehmen wir an, daß ein solcher Zug p existiert. Nach (7.2) ist dann einer der Punkte X_1 und X_2, z. B. X_2 von X^* verschieden. Es gibt dann einen X^*-Zug $p_0 = (X^* \ldots X_2)$. Nach (8.6) kann man jetzt (7.4) anwenden, woraus die Existenz eines a-Zuges $p_1 = (X^* \ldots {}^u X_1)$ oder $p_2 = (X^* \ldots {}^v X_2)$ folgt. Das widerspricht jedoch unserer Annahme, daß X_1 ein $\bar{u}$-Punkt und X_2 ein $\bar{v}$-Punkt ist.

Aus (8.7) folgt:

(8.8) *Zwei u-Punkte können nur durch u-Kanten verbunden sein.*

(8.9) *Ein unerreichbarer Punkt kann von den erreichbaren Punkten nur mit den α- und β-Punkten durch Kanten verbunden sein, und zwar mit einem α-Punkt nur durch eine α-Kante, mit einem β-Punkt nur durch eine β-Kante.*

Beweis. Es sei X ein unerreichbarer, X' ein erreichbarer Punkt und $[XxX']$ eine u-Kante. Offensichtlich ist $X' \neq X^*$. Ferner ist X' nicht $\bar{u}$-erreichbar. Ist nämlich X' $\bar{u}$-erreichbar, so gibt es einen X^*-Zug $p = (X^* \ldots \overset{\bar{u}}{} X')$. p kann X und demnach auch x nicht enthalten. Nach (6.2)(a) ist dann $p(X'xX)$ ein a-Zug und X erreichbar.

Wir zeigen, daß X' kein γ_u-Punkt sein kann. Nehmen wir an, daß X' ein γ_u-Punkt ist. Es gibt einen X^*-Zug p_1 mit dem Endpunkt X'. Es bezeichne p_2 den kürzesten Anfangsteil von p_1, dessen Endpunkt X' ist. Dann ist X' in p_2 einfach, die Endkante von p_2 eine u-Kante, und es kann weder X noch x in p_2 liegen. Demzufolge ist aber nach (8.4) und (6.2)(b), (c) $p_2(X'xX)$ ein X^*-Zug, und so wäre X erreichbar.

Es kann also X' nur ein u-Punkt sein.

§ 9. Erreichbare und unerreichbare Komponenten

(9.1) Es bezeichne A, B^* und C die Menge der α-, β- und γ-Punkte, Φ^* die Menge sämtlicher Punkte von Γ^*. Laut (8.3) ist $X^* \in B^*$. Es sei $B = B^* - \{X^*\}$. Mit diesen Mengen bestimmen wir das zu Γ^* bzw. Γ gehörige füllende Gewichtssystem

$$q^* = q^*(A, B^*, C) \quad \text{bzw.} \quad q = q(A, B, C).$$

Unser Ziel ist zu zeigen, daß q ein gesuchtes, der Bedingung (5.15)(1) genügendes System ist. In § 9 werden wir zu diesem Zwecke die Eigenschaften der C-Komponenten von Γ^* untersuchen.

(9.2) Im übrigen Teil dieses Paragraphen benützen wir nur die unter (8.1) erwähnten Eigenschaften von Γ^*, $\varkappa$, $\varkappa'$ und f^*.

Wir bezeichnen die C-Komponenten von Γ^*, d. h. die Komponenten des Teilgraphen $[C]$ von Γ^* mit $[C_i]$ $(i = 1, \ldots, m)$. C_i $(i = 1, \ldots, m)$ soll die Menge der Punkte von $[C_i]$ bedeuten. Nach (8.9) besteht eine nichtleere C-Komponente entweder aus lauter erreichbaren oder aus lauter unerreichbaren Punkten. Dementsprechend nennen wir eine C-Komponente entweder *erreichbar* oder *unerreichbar*. Im Falle $C = C_1 = \emptyset$ nennen wir $[C_1]$ unerreichbar. Nach (8.9) können wir ferner behaupten:

(9.3) *Berührt eine Kante eine C-Komponente, so ist der äußere Randpunkt der Kante (in Bezug der Komponente) entweder ein α- oder ein β-Punkt.*

Es gilt die Behauptung:

(9.4) *Ist $[C_i]$ eine nichtleere C-Komponente und $X \in C_i$, bezeichnet ferner $p = (X^* \ldots X'xX)$ einen solchen X^*-Zug, in dem $X' \notin C_i$ und x eine u-Kante ist, so ist X' ein $\bar{u}$-Punkt.*

Beweis. Nach (9.3) genügt es zu zeigen, daß X' kein u-Punkt ist. Ist X' ein u-Punkt und $X' \neq X^*$, so existiert die vorletzte Kante von p, und diese muß eine u-Kante sein. Dies widerspricht jedoch dem Satze (8.5). Ist X' ein u-Punkt und $X' = X^*$, so muß $u = \beta$ sein, was zu (7.1)(1) in Widerspruch steht.

Unter Beachtung von (9.4) wollen wir jede solche u-Kante, die die C-Komponente $[C_i]$ berührt und deren äußerer Randpunkt (in Bezug von $[C_i]$) ein $\bar{u}$-Punkt ist, eine *Eintrittskante* von $[C_i]$ nennen.

Aus (8.9) folgt:

(9.5) *Die unerreichbaren C-Komponenten besitzen keine Eintrittskante.*

Der folgende Satz bildet den Kernpunkt unserer ganzen Beweisführung:

Satz (9.6) *Jede erreichbare C-Komponente besitzt genau eine Eintrittskante.*

Wir zerlegen den Beweis dieses Satzes in mehrere Teile (von (9.7) bis (9.11)).

(9.7) Es sei $[C_i]$ eine beliebige erreichbare C-Komponente. Wir wollen $[C_i]$ im folgenden festhalten. Erst zeigen wir, daß $[C_i]$ eine Eintrittskante besitzt. Es sei $X \in C_i$. Da $X^* \notin C_i$, gibt es einen X^*-Zug p mit dem Endpunkt X. Es bezeichne $p' = (X^* \ldots X'xX'')$ den kürzesten Anfangsteil von p, dessen Endpunkt zu C_i gehört. Dann ist $X' \notin C_i$. Ist x eine u-Kante, so ist nach (9.4) X' ein $\bar{u}$-Punkt, und daher ist x eine Eintrittskante von $[C_i]$.

Es sei nun $[X_0 x_0 X_1]$ eine beliebige Eintrittskante von $[C_i]$ mit $X_0 \notin C_i$, $X_1 \in C_i$, es sei ferner x_0 eine u-Kante und X_0 ein $\bar{u}$-Punkt. Wir wollen im folgenden auch x_0 und u festhalten und zeigen, daß x_0 die einzige Eintrittskante von $[C_i]$ ist.

Wir wollen einen a-Zug p einen X_0*-Zug* nennen, wenn er folgende Bedingungen erfüllt: Es ist X_0 der Anfangspunkt, x_0 die Anfangskante von p, und mit Ausnahme von X_0 und x_0 gehören sämtliche Punkte und Kanten von p zu $[C_i]$. Ist $X \in C_i$ und gibt es einen X_0-Zug mit dem Endpunkt X, so sagen wir, daß X *in* $[C_i]$ *erreichbar* ist. Gibt es einen X_0-Zug mit dem Endpunkt X, dessen Endkante eine v-Kante ist, so heißt X in $[C_i]$ v*-erreichbar.*

Wir bemerken, daß jeder von X_0 verschiedene Punkt eines X_0-Zuges in $[C_i]$ erreichbar ist, und jeder nicht in $[C_i]$ liegende Punkt in $[C_i]$ unerreichbar ist.

(9.8) *Ist X v-erreichbar und in $[C_i]$ erreichbar, so ist er auch in $[C_i]$ v-erreichbar.*

Beweis. Es sei $X \in C_i$, $p = (X^* \dots {}^{v}X)$ ein X^*-Zug und p_0 ein X_0-Zug mit dem Endpunkt X. Ist die Endkante von p_0 eine v-Kante, so ist nichts zu beweisen. Es sei also $p_0 = (X_0 x_0 X_1 \dots {}^{\bar{v}}X)$. Da $X^* \notin C_i$ ist, gibt es einen kürzesten Endteil von p, dessen Anfangskante nicht zu $[C_i]$ gehört. Es sei dieser Teil $p_2 = (X_2 x_2 X_3 \dots {}^{v}X)$, und x_2 sei eine w-Kante. Außer X_2 und x_2 liegen sämtliche Punkte und Kanten von p_2 in $[C_i]$. Wendet man (9.4) auf denjenigen Anfangsteil von p an, dessen Endkante x_2 ist, so sieht man, daß X_2 ein $\bar{w}$-Punkt ist. Ist $x_2 = x_0$, so ist p_2 ein X_0-Zug, und dann ist der Beweis fertig. Nehmen wir an, daß $x_2 \neq x_0$ ist. Liegt X_0 auf p_2, so kann nur $X_0 = X_2$ bestehen, also ist X_0 ein Wechselpunkt von p_2. Wir können daher in jedem Falle (7.4) auf p_2 und p_0 anwenden. Demzufolge gibt es einen a-Zug $p_3 = (X_0 x_0 \dots {}^{w}X_2)$ oder $p_4 = (X_0 x_0 \dots {}^{v}X)$, und diese Züge enthalten nur solche Kanten, die entweder in p_2 oder in p_0 vorkommen. Nach (8.7) kann aber p_3 nicht existieren. Es existiert also p_4. Wir zeigen nun, daß p_4 ein X_0-Zug ist. Da von den Kanten von p_0 und p_2 nur x_0 und x_2 nicht zu $[C_i]$ gehören, genügt es zu beweisen, daß x_2 nicht in p_4 liegt. Nehmen wir an, daß x_2 in p_4 liegt. Von den Kanten von p_4 können nur x_0 und x_2 zu X_2 inzident sein. Ist $X_2 = X_0$, so sind wegen $x_0 \neq x_2$ genau zwei Kanten von p_4 mit X_2 inzident. Dies ist jedoch unmöglich, da X_0 der Anfangspunkt, jedoch nicht der Endpunkt von p_4 ist. Ist $X_2 \neq X_0$, so ist von den Kanten von p_4 genau eine Kante zu X_2 inzident. Dies ist aber wieder unmöglich, da jetzt X_2 kein Randpunkt von p_4 sein kann. Es ist also p_4 tatsächlich ein X_0-Zug, und daher ist X in $[C_i]$ v-erreichbar.

(9.9) *Ist X' ein in $[C_i]$ erreichbarer, X ein in $[C_i]$ nicht erreichbarer Punkt und ist x eine von x_0 verschiedene $X'X$-Kante, so existiert ein a-Zug $p = (X_0 x_0 \dots x X)$, dessen sämtliche, von x_0 und x verschiedene Kanten zu $[C_i]$ gehören.*

Beweis. Es sei x eine v-Kante. Liegt X auf einem X_0-Zug, so kann nur $X = X_0$ bestehen, und deshalb kann x zu keinem X_0-Zug gehören.

(I) Nehmen wir erst an, daß ein X_0-Zug $p_0 = (X_0 x_0 \dots {}^{\bar{v}}X')$ existiert. Man kann dann (6.2)(a) auf die a-Züge p_0 und $p' = (X' x X)$ anwenden, demzufolge ist $p = p_0 p'$ ein gesuchter a-Zug.

(II) Es sei jetzt X' in $[C_i]$ nicht $\bar{v}$-erreichbar. Es gibt einen X_0-Zug p_1 mit dem Endpunkt X'. Es bezeichne p_2 den kürzesten Anfangsteil von p_1,

dessen Endpunkt X' ist. X' ist im X_0-Zug p_2 einfach, und die Endkante von p_2 ist eine v-Kante. Nach (9.8) kann jedoch X' jetzt nicht $\bar{v}$-erreichbar sein, daher muß X' ein γ_v-Punkt sein. Demzufolge ist nach (8.4) und (6.2)(b), (c) $p = p_2 p'$ ein gesuchter a-Zug.

(9.10) *Jeder Punkt von C_i ist in $[C_i]$ erreichbar.*

Beweis. Der Punkt X_1 der Kante x_0 ist offensichtlich in $[C_i]$ erreichbar. Nehmen wir an, daß in $[C_i]$ Punkte existieren, die in $[C_i]$ nicht erreichbar sind. Da $[C_i]$ zusammenhängend ist, gibt es dann in $[C_i]$ eine Kante x, die einen in $[C_i]$ erreichbaren Punkt X' mit einem in $[C_i]$ nicht erreichbaren Punkt X von C_i verbindet. Nach (9.9) existiert jedoch dann ein a-Zug $p = (X_0 x_0 \ldots x X)$, dessen sämtliche Kanten außer x_0 zu $[C_i]$ gehören. Es ist also X in $[C_i]$ erreichbar, was ein Widerspruch ist.

(9.11) Nehmen wir nun endlich an, daß eine von x_0 verschiedene Eintrittskante von $[C_i]$ existiert. Es sei $[X x X']$ eine solche Kante mit $X \notin C_i$, $X' \in C_i$ und x sei eine v-Kante, X' ein $\bar{v}$-Punkt. Nach (9.10) ist X' in $[C_i]$ erreichbar. Nach (9.9) gibt es dann einen a-Zug $p = (X_0 x_0 \ldots x X)$. Das widerspricht jedoch (8.7). Damit haben wir den Beweis von (9.6) beendet.

Nach (9.6) können wir nach der Art der Eintrittskanten die erreichbaren C-Komponenten in zwei Klassen teilen. Wir nennen eine erreichbare Komponente α- oder *β-erreichbar*, je nachdem ob die Eintrittskante eine α- oder β-Kante ist. Wir können dann nach (9.3), (8.9) und (9.6) zusammenfassend behaupten:

(9.12) *Ist $[C_i]$ eine beliebige C-Komponente, so sind die Kanten, die $[C_i]$ berühren, entweder AC_i- oder B^*C_i-Kanten, und zwar sind — mit Ausnahme der eventuell vorhandenen einzigen Eintrittskante — sämtliche AC_i-Kanten α-Kanten, sämtliche B^*C_i-Kanten β-Kanten.*

(1) *Ist $[C_i]$ unerreichbar, so gibt es keine Ausnahme.*

(2) *Ist $[C_i]$ α-erreichbar, so ist eine B^*C_i-Kante α-Kante.*

(3) *Ist $[C_i]$ β-erreichbar, so ist eine AC_i-Kante β-Kante.*

(9.13) *Bezeichnet man die Anzahl der α- bzw. β-erreichbaren C-Komponenten mit σ_α bzw. σ_β, so gibt es unter den AC-Kanten genau $\nu(A, C) - \sigma_\beta$, unter den B^*C-Kanten genau σ_α α-Kanten.*

Bezüglich q^* haben wir eine C-Komponente $[C_i]$ je nachdem gerade bzw. ungerade genannt, ob die Zahl $\varkappa(C_i) + \nu(A, C_i)$ gerade oder ungerade ist (s. (3.2)). Es gilt nun:

(9.14) *Es sind sämtliche erreichbaren C-Komponenten ungerade, sämtliche unerreichbaren gerade.*

Beweis. Betrachten wir eine beliebige C-Komponente $[C_i]$ $(1 \leqq i \leqq m)$. Es sei μ_i bzw. ν_i die Anzahl derjenigen α-Kanten, die $[C_i]$ berühren bzw. die in $[C_i]$ liegen. Da jede Kante zu genau zwei Punkten inzident ist, und die betrachteten α-Kanten außerhalb von $[C_i]$ genau μ_i, in den Punkten von $[C_i]$ genau $\sum\limits_{X \in C_i} (f^*, X)$ Inzidenzen hervorrufen, gilt

$$(1) \qquad 2(\mu_i + \nu_i) = \mu_i + \sum_{X \in C_i} (f^*, X).$$

Andernfalls ist nach (7.1)(3), (4) für jedes X die Zahl $(f^*, X) - \varkappa(X)$ gerade, und demzufolge ist auch

$$\sum_{X \in C_i} (f^*, X) - \varkappa(C_i) = \sum_{X \in C_i} ((f^*, X) - \varkappa(X))$$

gerade. Daraus und aus (1) folgt

$$(2) \qquad \mu_i + \varkappa(C_i) \equiv 0 \pmod 2.$$

Nach (9.12) ist aber

$$\mu_i = \nu(A, C_i) + \varepsilon_i,$$

wo $\varepsilon_i = 0$, 1 oder -1 ist, je nachdem ob $[C_i]$ unerreichbar, α- oder β-erreichbar ist. Dies ergibt zusammen mit (2) die Richtigkeit unserer Behauptung.

§ 10. Bestimmung von $S(q)$

Wir wollen bezüglich des Wertes $S(q^*)$ des unter (9.1) definierten Gewichtssystems q^* folgende Behauptung beweisen:

(10.1) *Es gilt* $2S(q^*) = \varkappa(\Phi^*)$.

Beweis. Es ist nach (3.2)(2)

$$S(q^*) = \varkappa(B^*) + \varkappa'(B^*) + \nu(A, A) + \frac{1}{2}(\varkappa(C) + \nu(A, C) - \tau),$$

wo τ die Anzahl der ungeraden C-Komponenten bedeutet.

Da $\varkappa(\Phi^*) = \varkappa(A) + \varkappa(B^*) + \varkappa(C)$ ist, ist (10.1) der folgenden Behauptung gleichwertig:

$$(1) \qquad \varkappa(A) = \varkappa(B^*) + 2\varkappa'(B^*) + 2\nu(A, A) + \nu(A, C) - \tau.$$

Um dies zu beweisen, werden wir die Anzahl ν_α derjenigen α-Kanten, die den Teilgraphen $[B^*]$ berühren, auf zweierlei Weisen ausdrücken.

(I) Ist X ein β-Punkt, so folgt im Falle $\varkappa'(X) > 0$ aus (8.2) und (7.1)(3), im Falle $\varkappa'(X) = 0$ aus (7.1)(3), (4) die Gleichung $(f^*, X) = \varkappa(X) + 2\varkappa'(X)$. Nach (8.8) können wir dann behaupten

$$(2) \qquad \nu_\alpha = \varkappa(B^*) + 2\varkappa'(B^*).$$

(II) Aus (7.1)(4) und (8.2) folgt, daß zu jedem α-Punkt X genau $\varkappa(X)$ α-Kanten inzident sind. Nach (8.8) ist jede AA-Kante eine α-Kante. Wir können daher nach (9.13) behaupten: unter den AB^*-Kanten kommen genau

$$\varkappa(A) - 2\nu(A, A) - (\nu(A, C) - \sigma_\beta)$$

α-Kanten vor. Daraus erhalten wir nach (9.13) für ν_α den Ausdruck

$$\nu_\alpha = \varkappa(A) - 2\nu(A, A) - \nu(A, C) + \sigma_\alpha + \sigma_\beta. \tag{3}$$

Es ist aber nach (9.14) $\sigma_\alpha + \sigma_\beta = \tau$, und so ergibt sich aus (2) und (3) die Gleichung (1).

(10.2) Um den Wert $S(q)$ zu erhalten, nehmen wir außer $B^* = B \cup \{X^*\}$ und (7.1)(a), (b) auch in Betracht, daß $\nu_\Gamma(A, A) = \nu(A, A)$, $\nu_\Gamma(A, C) = \nu(A, C)$ ist und die C-Komponenten von Γ mit denen von Γ^* identisch sind, sowie daß diese Komponenten in Γ und in Γ^* gleichzeitig gerade oder ungerade sind. Wir erhalten so die Gleichungen

$$S(q^*) = S(q) + \tilde{\delta}_{\min} \quad \text{und} \quad \varkappa(\Phi^*) = \varkappa(\Phi) + \tilde{\delta}_{\min}.$$

Diese ergeben jedoch zusammen mit (10.1) für $S(q)$ den Wert

$$S(q) = \frac{\varkappa(\Phi) - \tilde{\delta}_{\min}}{2}.$$

Nach (5.15) haben wir damit den Beweis des Hauptsatzes (3.3) beendet.

III. SPEZIELLE GEWICHTSSYSTEME UND KAPAZITÄTSFUNKTIONEN

§ 11. Gewichtssysteme mit besonderen Eigenschaften

In diesem Paragraphen wollen wir zeigen, daß man sich bei der Bestimmung des minimalen Wertes von $S(q)$ auf Gewichtssysteme mit besonderen Eigenschaften beschränken kann. Dies wird uns bei gewissen speziellen Kapazitätsfunktionen es ermöglichen, unserem Hauptsatz eine übersichtlichere Fassung zu geben.

(11.1) Es seien in den Punkten des Graphen Γ die Kapazitätsfunktionen $\varkappa(X)$ und $\varkappa'(X)$ definiert,[6] und es sei $q = q(A, B, C)$ ein beliebiges Gewichtssystem mit $A \neq \emptyset$, ferner sei $X \in A$. Wir wollen untersuchen, welche Änderung $S(q)$ bei der Verlegung von X nach C erfährt. Es sei

$$A' = A - \{X\}, \quad C' = C \cup \{X\}, \quad q' = q'(A', B, C').$$

Es gilt folgende Behauptung:

[6] Von hier an beziehen sich unsere Begriffe und Bezeichnungen wieder auf den mit Γ bezeichneten Graphen.

(11.2) *Ist* $\nu(X, A) \geqq \varkappa(X)-1$, *so ist* $S(q') \leqq S(q)$.

Beweis. Es ist $\varkappa(C') = \varkappa(C) + \varkappa(X)$, $\nu(A', A') = \nu(A, A) - \nu(X, A)$ und $\nu(A', C') = \nu(A, C) + \nu(X, A) - \nu(X, C)$. Daraus folgt nach (3 2)(2)

$$S(q') = S(q) + \frac{1}{2}(\varkappa(X) - \nu(X, A) - \nu(X, C) + \tau_q - \tau_{q'}).$$

Ist $\nu(X, C) = 0$, so sind sämtliche C-Komponenten auch C'-Komponenten, und außer diesen gibt es nur noch eine C'-Komponente: X. Mithin ist $\tau_{q'} = \tau_q$ oder $\tau_{q'} = \tau_q + 1$, je nachdem ob $\varkappa(X) + \nu(X, A)$ gerade oder ungerade ist. Im ersten Falle folgt daraus, da $\nu(X, A) = \varkappa(X) - 1$ nicht eintreten kann, die Behauptung $S(q') \leqq S(q)$. Im zweiten Falle ist die gleiche Behauptung offensichtlich richtig.

Nehmen wir nun an, daß $\nu(X, C) > 0$ ist. Es seien $[C_i]$ $(i = 1, \ldots, m)$ die C-Komponenten, und von diesen seien $[C_i]$ $(i = j+1, \ldots, m)$ diejenigen, für welche $\nu(C_i, A) > 0$ gilt $(0 \leqq j < m)$. Dann sind die C'-Komponenten: $[C'_{j+1}]$ mit $C'_{j+1} = (\bigcup_{i=j+1}^{m} C_i) \cup \{X\}$ und, falls $j > 0$ ist, $[C'_i] = [C_i]$ $(i = 1, \ldots, j)$.

Es bezeichne ferner σ die Anzahl der ungeraden Komponenten unter $[C_{j+1}], \ldots, [C_m]$, und es sei $\varepsilon = 0$ oder 1, je nachdem ob $[C'_{j+1}]$ gerade oder ungerade ist. Es gilt dann

$$\tau_q - \tau_{q'} = \sigma - \varepsilon$$

und

$$S(q') = S(q) - \frac{1}{2}(\nu(X, A) - \varkappa(X) + \nu(X, C) - \sigma + \varepsilon).$$

Es besteht ferner $\nu(X, C) \geqq m - j \geqq \sigma$.

Gilt nun $\nu(X, A) \geqq \varkappa(X)$ oder $\nu(X, C) > \sigma$, so ist offensichtlich $S(q') \leqq S(q)$.

Ist $\nu(X, A) = \varkappa(X) - 1$ und $\nu(X, C) = \sigma$, so zeigen wir, daß $\varepsilon = 1$ ist, woraus wieder $S(q') \leqq S(q)$ folgen wird. In diesem Falle muß nämlich $\nu(X, C_i) = 1$ und $\varkappa(C_i) + \nu(A, C_i) \equiv 1 \pmod 2$ für jedes $i = j+1, \ldots, m$ bestehen und deshalb ist die Zahl

$$\varkappa(C'_{j+1}) + \nu(A', C'_{j+1}) = \varkappa(X) + \nu(X, A) + \sum_{i=j+1}^{m} (\varkappa(C_i) + \nu(A, C_i) - 1)$$

ungerade.

(11.3) Nehmen wir jetzt von $q = q(A, B, C)$ an, daß $C \neq \varnothing$ ist, und es sei $X \in C$. Das System q' definieren wir folgendermaßen: Es sei

$$B' = B \cup \{X\}, \quad C' = C - \{X\} \quad \text{und} \quad q' = q'(A, B', C').$$

Es gilt dann die Behauptung:

(11.4) *Ist* $\nu(X, A) \geqq \varkappa(X) + 2\varkappa'(X)$, *so ist* $S(q') \leqq S(q)$.

BEWEIS. Eine einfache Rechnung zeigt, daß

$$S(q') = S(q) - \frac{1}{2}\left(\nu(A, X) - (\varkappa(X) + 2\varkappa'(X)) + \tau_{q'} - \tau_q\right)$$

ist. Es sei $[C_1]$ diejenige C-Komponente, die X enthält, und es sei $C^* = C_1 - \{X\}$. Die Gesamtheit der C'-Komponenten besteht aus sämtlichen von C_1 verschiedenen C-Komponenten sowie (falls $C^* \neq \emptyset$) aus den Komponenten des Teilgraphen $[C^*]$. Mithin ist $\tau_{q'} \geqq \tau_q - 1$, und die Gleichheit kann hier nur dann eintreten, wenn $[C_1]$ ungerade ist und sämtliche Komponenten von $[C^*]$ gerade sind oder $C^* = \emptyset$ ist. In diesen Fällen muß jedoch $\varkappa(X) + \nu(A, X)$ ungerade sein, und so kann in $\nu(A, X) \geqq \varkappa(X) + 2\varkappa'(X)$ das Gleichheitszeichen nicht gelten. Hieraus sehen wir, daß in jedem Falle $S(q') \leqq S(q)$ gilt.

(11.5) Es bezeichne Q_0 die Menge jener Gewichtssysteme $q = q(A, B, C)$, welche folgende Eigenschaften besitzen:

(a) A enthält, falls es nichtleer ist, nur solche Punkte X, die der Ungleichung $\nu(X, A) < \varkappa(X) - 1$ genügen.

(b) C enthält, falls es nichtleer ist, nur solche Punkte X, die der Bedingung $\nu(X, A) < \varkappa(X) + 2\varkappa'(X)$ genügen.

Nach (11.2) und (11.4) können wir dann behaupten:

$$\min_{q \in Q_0} S(q) = S_{\min}. \tag{1}$$

Wir nennen die Punkte der Menge $E \subset \Phi$ (in Γ) *unabhängig*, wenn im Falle $\nu(E) > 1$ je zwei von ihnen durch keine Kante (von Γ) verbunden sind. Dann kann man folgende Behauptungen machen:

(11.6) *Ist für jedes* X $\varkappa(X) + \varkappa'(X) = 1$, *so ist in jedem System* $q = q(A, B, C)$ *von* Q_0 *die Menge* A *leer.*

(11.7) *Ist für jedes* X $\varkappa(X) + 2\varkappa'(X) = 2$, *so besitzt jedes System* $q = q(A, B, C)$ *von* Q_0 *die Eigenschaften:*

(1) *Die* A*-Punkte sind unabhängig, und für jedes* X *von* A *gilt* $\varkappa(X) = 2$.

(2) *Zu jedem* C*-Punkt ist höchstens eine* AC*-Kante inzident.*

§ 12. Haupt- und Nebenpunkte. Hauptwege

(12.1) Wir wollen einige spezielle Kapazitätsfunktionen betrachten, bei denen unser Hauptsatz bzw. einige nachfolgende Sätze eine übersichtlichere Formulierung zulassen. Nehmen wir folgende Bedingungen:

(a) Für jedes X mit $\varkappa(X) > 0$ ist $\varkappa'(X) = 0$.

(b) Für jedes X mit $\varkappa(X) = 0$ ist $\varkappa'(X) = 1$.

Genügen die Funktionen $\varkappa$ und $\varkappa'$ diesen Bedingungen, so ist $\varkappa'$ durch $\varkappa$ eindeutig bestimmt, und wir nennen in diesem Falle $\varkappa$ und $\varkappa'$ zueinander *komplementär*. Ferner nennen wir jetzt die Punkte X mit $\varkappa(X) > 0$ *Hauptpunkte*, diejenigen mit $\varkappa(X) = 0$ *Nebenpunkte* und diejenigen Bogen, deren Randpunkte bzw. innere Punkte Haupt- bzw. Nebenpunkte sind, *Hauptbogen*. Die Bogen eines aufnehmbaren Bogensystems sind jetzt alle Hauptbogen, und diese können nur Randpunkte gemein haben.

Eine weitere beachtenswerte Spezialisierung erhalten wir durch die Annahme, daß bei komplementären Kapazitäten $\varkappa$ in jedem Hauptpunkt den gleichen Wert σ annimmt. Um ein solches Paar von Kapazitätsfunktionen anzugeben, genügt es, irgendwelche Punkte von Γ als Hauptpunkte auszeichnen, die übrigen als Nebenpunkte betrachten und den Wert σ vorschreiben. (Die Menge der Haupt- bzw. Nebenpunkte kann auch leer sein!)

Von hier an werden wir uns im II. Abschnitt nur mit den letzterwähnten Kapazitätsfunktionen beschäftigen, und zwar nur im Falle $\sigma = 1$.

(12.2) Es seien nun in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Die übrigen Punkte von Γ werden wir dann immer — ohne dies ausdrücklich zu betonen — Nebenpunkte nennen. Die Funktionen $\varkappa(X)$ und $\varkappa'(X)$ seien durch folgende Bedingungen definiert: Für jeden Hauptpunkt sei $\varkappa(X) = 1$ und $\varkappa'(X) = 0$, für jeden Nebenpunkt $\varkappa(X) = 0$ und $\varkappa'(X) = 1$. Wir bezeichnen die Anzahl der Haupt- bzw. Nebenpunkte in einer beliebigen Teilmenge E von Φ mit $\nu_h(E)$ bzw. $\nu_n(E)$.

Die aufnehmbaren Bogen fallen jetzt mit den *Hauptwegen* zusammen. Ein Hauptweg ist ein solcher Weg, dessen Randpunkte bzw. inneren Punkte Haupt- bzw. Nebenpunkte sind. Ein aufnehmbares Bogensystem besteht aus *unabhängigen* Hauptwegen, d. h. aus solchen Hauptwegen, die paarweise keinen gemeinsamen Punkt enthalten. $\nu_{\max}$ gibt jetzt *die maximale Anzahl der unabhängigen Hauptwege* an.

Betrachten wir nun den Wert $S_{\min}$. Nach (11.5)(1) und (11.6) genügt es bei der Bestimmung dieses Wertes, nur solche Systeme $q = q(A, B, C)$ zu betrachten, in denen A leer ist. Solche Systeme sind durch die Angabe der einzigen, beliebig wählbaren Teilmenge B von Φ bestimmt. Wir setzen daher für solche q

$$S(q) = S'(B) \quad \text{und} \quad \tau_q = \tau(B) \quad (\bar{B} = \Phi - B).$$

$\tau(B)$ gibt die Anzahl der ungeraden, d. h. derjenigen Komponenten von $[\bar{B}]$, die eine ungerade Anzahl von Hauptpunkten enthalten. Bezeichnen $[B_i]$

$(i=1,\ldots,m)$ die Komponenten von $[\bar{B}]$, so gilt nach (3.2)(1), (2), (11.6) und (11.5)

$$(1)\qquad S'(B)=\nu(B)+\sum_{i=1}^{m}\left|\frac{1}{2}\nu_h(\bar{B}_i)\right|=\nu(B)+\frac{1}{2}(\nu_h(B)-\tau(\bar{B}))$$

und

$$\min_{B\subseteq\Phi} S'(B)=S_{\min}.$$

Unseren Hauptsatz (3.3) kann man jetzt folgendermaßen formulieren:

Satz (12.3) *Sind in Γ irgendwelche Punkte als Hauptpunkte ausgewählt, so ist die maximale Anzahl der unabhängigen Hauptwege dem Wert* $\min\limits_{B\subseteq\Phi} S'(B)$ *gleich.*

Wegen späteren Anwendungen wollen wir noch folgende Behauptung beweisen:

(12.4) *Gilt für die Teilmenge B von Φ die Ungleichung $S'(B)<S'(\emptyset)$, so ist $\tau(\bar{B})\geqq 2$.*

Beweis. Es ist $S'(\emptyset)=\frac{1}{2}(\nu_h(\Phi)-\tau(\Phi))$. Wir erhalten also aus $S'(B)<S'(\emptyset)$ die Ungleichung $\tau(\bar{B})>\nu(B)+\nu_h(B)+\tau(\Phi)$. Da jetzt $B\neq\emptyset$ gilt, besteht $\tau(\bar{B})\geqq 2$.

(12.5) Bisher haben wir von den Gewichten der füllenden Gewichtssysteme verlangt, daß sie jeden Bogen des Graphen füllen (s. (2.2)). Lassen wir diese Forderung fallen und verlangen nur, daß sämtliche Hauptwege gefüllt werden, so sind bei einem System $q=q(\emptyset, B, \bar{B})$ sämtliche in Nebenpunkten liegenden halben Gewichte zu der Füllung überflüssig. Es ist jetzt naheliegend, eine neue Art von Gewichtssystemen zu betrachten. Diese Systeme, die wir Systeme *zweiter Art* nennen und mit dem Buchstaben r bezeichnen wollen, definieren wir folgendermaßen: In r sollen nur die Punkte Gewichte erhalten, und zwar sollen wieder nur die Gewichte 0, 1 und 1/2 vorkommen. Einen Hauptweg w nennen wir jetzt durch r dann gefüllt, wenn w einen Punkt mit dem Gewicht 1 oder zwei Punkte (diese müssen nicht die Randpunkte von w sein) mit halben Gewichten enthält. r heißt *füllend*, wenn jeder Hauptweg von Γ durch r gefüllt ist, oder wenn in Γ kein Hauptweg existiert. Die Menge dieser füllenden Gewichtssysteme bezeichnen wir mit R. R ist offensichtlich nichtleer.

Den *Wert* von r definieren wir gleichfalls von neuem. Es bezeichne B die Menge derjenigen Punkte, die in r das Gewicht 1 erhalten und $\nu_{1/2}(E)$ die Anzahl derjenigen Punkte einer beliebigen Teilmenge E von Φ, die in r das Gewicht 1/2 haben. Unter Beachtung von (12.2) und der oben Gesagten

sei der Wert von r

(1) $$S''(r) = \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2}\nu_{1/2}(\bar{B}_i)\right],$$

wo $[\bar{B}_i]$ $(i=1,\ldots,m)$ die Komponenten von $[\bar{B}]$ bedeuten $(\bar{B} = \Phi - B)$.

Wir wollen nun zeigen, daß die Behauptung von (12. 3) auch mit den Gewichtssystemen zweiter Art gültig bleibt:

SATZ (12. 6) *Sind in Γ irgendwelche Punkte als Hauptpunkte ausgewählt, so ist die maximale Anzahl der unabhängigen Hauptwege dem Wert* $\min\limits_{r \in R} S''(r)$ *gleich.*

BEWEIS. (I) Es sei B eine beliebige Teilmenge von Φ, und $[\bar{B}_i]$ $(i=1,\ldots,m)$ seien die Komponenten von $[\bar{B}]$. Ferner bezeichne r_1 jenes Gewichtssystem zweiter Art, in dem die Punkte von B das Gewicht 1, die Hauptpunkte von $\bar{B}$ das Gewicht 1 2, die Nebenpunkte von B das Gewicht 0 erhalten. Es ist dann $r_1 \in R$ und es gilt $\nu_{1/2}(B_i) = \nu_{r_1}(B_i)$ $(i=1,\ldots,m)$. Nach (12. 2) (1) und (12. 5) (1) ist dann $S''(r_1) = S'(B)$, und so ist laut (12. 3) $\min\limits_{r \in R} S''(r) \leqq \nu_{\max}$.

(II) Es sei jetzt r ein beliebiges Element von R. Es bezeichne ferner B die Menge der Punkte, die in r das Gewicht 1 enthalten, und $[\bar{B}_i]$ $(i=1,\ldots,m)$ seien die Komponenten von $[\bar{B}]$. Endlich sei W ein beliebiges System unabhängiger Hauptwege. Die Anzahl der Wege von W, die einen B-Punkt enthalten, ist $\leq \nu(B)$. Ein Weg von W, der keinen B-Punkt enthält, muß ganz in einer der Komponenten $[\bar{B}_i]$ liegen. In einem $[\bar{B}_i]$ können aber höchstens $\left[\frac{1}{2}\nu_{1/2}(\bar{B}_i)\right]$ unabhängige Hauptwege liegen. Es folgt daraus nach (12. 5) (1), daß $\nu(W) \leq S''(r)$ ist. Wir gelangen so zu $\nu_{\max} = \min\limits_{r \in R} S''(r)$, und damit ist unser Beweis beendet.

§ 13. Trennungssätze

(13. 1) Es sei Γ wieder ein Graph, in dem irgendwelche Punkte als Hauptpunkte ausgezeichnet sind. Die einfachsten füllenden Gewichtssysteme zweiter Art sind offensichtlich jene, bei denen nur die Gewichte 1 und 0 vorkommen. Ein solches System ist durch die Menge D der Punkte mit positiven Gewichten vollständig bestimmt und so beschaffen, daß jeder Hauptweg von Γ einen D-Punkt enthält. Wir können hier statt der Füllung der Hauptwege auch von der Trennung der Hauptpunkte sprechen, da man von keinem Hauptpunkt zu einem anderen ohne Berührung eines D-Punktes gelangen kann. (Auch die Randpunkte der Hauptwege können an der Trennung teilnehmen!)

Wir nennen nun eine Teilmenge D von Φ *trennend*, wenn jeder Hauptweg von Γ mindestens einen D-Punkt enthält (oder wenn Γ keinen Hauptweg besitzt). T bezeichne die Menge der trennenden Teilmengen (T ist nichtleer). Es sei ferner

$$\pi_{\min} = \min_{D \in T} \nu(D).$$

Die Zahl $\pi_{\min}$ existiert und bedeutet *die minimale Anzahl der trennenden Punkte.*

Es stellt sich dann die Frage: Was für ein Zusammenhang besteht zwischen $\pi_{\min}$ und $\nu_{\max}$? Man kann als Antwort im allgemeinen keine genaue Gleichung angeben, sondern muß sich mit Ungleichungen begnügen.

Es besteht folgender

SATZ (13. 2) *Sind in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet, so gilt*

$$\pi_{\min} \leqq 2\nu_{\max}.$$

BEWEIS. Nach (12. 3) gibt es ein $B \subseteq \Phi$ mit

$$S'(B) = \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2}\nu_h(\bar{B}_i)\right] = \nu_{\max} \qquad (\bar{B} = \Phi - B), \tag{1}$$

wo $[\bar{B}_i]$ $(i = 1, \ldots, m)$ die Komponenten von $[\bar{B}]$ sind. Wir definieren eine Menge D wie folgt: Wir wählen in jedem $\bar{B}_i$, das Hauptpunkte enthält, je einen Hauptpunkt aus. Es soll nun D aus sämtlichen Punkten von B sowie aus sämtlichen Hauptpunkten von $\bar{B}$, mit Ausnahme der ausgewählten, bestehen. Es ist klar, daß dieses D trennend ist, und es gilt

$$\nu(D) \leqq \nu(B) + 2\sum_{i=1}^{m}\left[\frac{1}{2}\nu_h(\bar{B}_i)\right]. \tag{2}$$

Aus (1) und (2) folgt

$$\nu(D) \leqq 2\nu_{\max} - \nu(B), \tag{3}$$

und dies ergibt $\pi_{\min} \leqq 2\nu_{\max}$.

BEMERKUNGEN (1) Satz (13. 2) kann ohne weitere Voraussetzungen nicht verschärft werden. Dies zeigt folgendes Beispiel: Γ bestehe aus n isolierten „Dreiecken" und jeder Punkt von Γ sei Hauptpunkt. Dann ist $\nu_{\max} = n$, $\pi_{\min} = 2n$.

(2) Ist Γ ein paarer Graph und sind sämtliche Punkte von Γ Hauptpunkte, so kann man durch eine einfache Modifizierung des Beweises von (13. 2) zu der Ungleichung $\pi_{\min} \leqq \nu_{\max}$ gelangen. Da anderseits offensichtlich $\pi_{\min} \geqq \nu_{\max}$ besteht, erhalten wir $\pi_{\min} = \nu_{\max}$. Diese Gleichung ist mit der Behauptung des Königschen Satzes (1) unserer Einleitung identisch.

(13.3) Der Graph in der vorangehenden Bemerkung (1) ist nicht zusammenhängend. Nimmt man an, daß der Graph zusammenhängend ist, bzw. daß der „Zusammenhangsgrad" des Graphen einen gewissen Wert erreicht, so kann man den Satz (13.2) verschärfen.

Wir sagen: Die verschiedenen Punkte X' und X'' sind durch die von X' und X'' verschiedenen Punkte $X_1, \ldots, X_j$ $(j \geqq 1)$ *getrennt*, wenn jeder Weg von Γ, der X' mit X'' verbindet, mindestens einen der Punkte $X_1, \ldots, X_j$ enthält. Die Aussage „*Γ ist bezüglich der Hauptpunkte η-fach zusammenhängend*" soll folgendes bedeuten:

Im Falle $\eta = 1$: Höchstens eine der Komponenten von Γ enthält Hauptpunkte. In diesem Falle werden wir auch Γ bezüglich der Hauptpunkte zusammenhängend nennen.

Im Falle $\eta \geqq 2$: Γ ist bezüglich der Hauptpunkte zusammenhängend, und je zwei Hauptpunkte X' und X'' können durch weniger als η, von X' und X'' verschiedenen Punkten nicht getrennt werden.

Aus dieser Erklärung folgt: Ist Γ bezüglich der Hauptpunkte η-fach zusammenhängend $(\eta > 1)$, so ist er auch $(\eta - 1)$-fach zusammenhängend.

Nun wollen wir unter Beachtung von (4.2) (2) folgenden Satz aussprechen:

SATZ (13.4) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Ist dann Γ bezüglich der Hauptpunkte η-fach zusammenhängend $(\eta \geqq 1)$ und ist $\nu_{\max} < \left[\frac{1}{2}\nu_h(\Phi)\right]$, so ist*

$$\pi_{\min} \leqq 2\nu_{\max} - \eta.$$

BEWEIS. Es sei B dieselbe Menge wie im Beweis von (13.2). Da Γ bezüglich der Hauptpunkte zusammenhängend ist, ist $S'(\emptyset) = \left[\frac{1}{2}\nu_h(\Phi)\right]$, und so gilt $S'(B) < S'(\emptyset)$. Dann ist aber nach (12.4) $\tau(\bar{B}) \geqq 2$. Jede ungerade $\bar{B}$-Komponente enthält jedoch einen Hauptpunkt, und so existieren in $\bar{B}$ zwei Hauptpunkte, die durch die Punkte von B getrennt werden. Es gilt daher $\nu(B) \geqq \eta$, und so folgt laut der Ungleichung (3) des Beweises von (13.2) die Behauptung $\pi_{\min} \leqq 2\nu_{\max} - \eta$.

BEMERKUNGEN. (1) Gilt nicht $\nu_{\max} < \left[\frac{1}{2}\nu_h(\Phi)\right]$, so ist nach (4.2) (2) $\nu_{\max} = \left[\frac{1}{2}\nu_h(\Phi)\right]$. In diesem Falle kann — wie das folgende Beispiel zeigt — $\pi_{\min} > 2\nu_{\max} - \eta$ eintreten.

Es sei $\Phi = \{X_1, \ldots, X_n, X'_1, \ldots, X'_\eta\}$ $(\eta \geqq 2,\ \eta < n < 2\eta)$ und Γ soll je eine $X_iX'_j$-Kante $(i = 1, \ldots, n;\ j = 1, \ldots, \eta)$ enthalten, jedoch keine andere

Kanten. $X_1, \ldots, X_n$ seien die Hauptpunkte, $X_1', \ldots, X_\eta'$ die Nebenpunkte. Dann ist Γ bezüglich der Hauptpunkte η-fach zusammenhängend, $\nu_{\max} = \left[\frac{n}{2}\right] = \left[\frac{1}{2}\nu_h(\Phi)\right]$ und $\pi_{\min} = \eta$.

(2) Ohne weitere Bedingungen kann man auch (13. 4) nicht verschärfen. Das zeigt folgendes Beispiel: Es sei

$$A = \{X_1, \ldots, X_n\} \quad B = \{X_1', \ldots, X_\eta'\} \qquad (2 \leqq 2\eta \leqq n),$$
$$C_i = \{X_1^i, X_2^i, X_3^i\} \quad (i = 1, \ldots, l;\ l \geqq 1),$$

und es seien A, B und sämtliche C_i paarweise elementenfremd. Φ bestehe aus sämtlichen Punkten der Mengen A, B und C_i $(i = 1, \ldots, l)$, die Menge der Kanten von Γ

aus je einer $X_i X_j'$-Kante $(i = 1, \ldots, n;\ j = 1, \ldots, \eta)$,
aus je einer $X_k^i X_j'$-Kante $(i = 1, \ldots, l;\ j = 1, \ldots, \eta;\ k = 1, 2, 3)$ und
aus je einer $X_j^i X_k^i$-Kante $(i = 1, \ldots, l;\ j, k = 1, 2, 3;\ j \neq k)$.

Die A-Punkte und die C_i-Punkte seien die Hauptpunkte, die B-Punkte die Nebenpunkte. Es ist dann Γ bezüglich der Hauptpunkte η-fach zusammenhängend, $\nu_{\max} = \eta + l$, $\pi_{\min} = \eta + 2l$. Es gilt daher $\pi_{\min} = 2\nu_{\max} - \eta$.

(3) Wir vermuten folgende Verallgemeinerung von (13. 2):

Sind in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet, und ist die maximale Anzahl solcher Hauptwege, die paarweise keinen Nebenpunkt gemeinsam haben, gleich $\nu'_{\max}$, so ist die minimale Anzahl der derart auswählbaren Nebenpunkte, daß jeder Hauptweg mindestens einen der ausgewählten Punkte enthält, nicht größer als $2\nu'_{\max}$.

Dieser Satz scheint tieferliegend zu sein als (13. 2). Man könnte natürlich einen Beweis dieser Vermutung erhalten, wenn man einen unserem Hauptsatz ähnlichen Satz über Hauptwege hätte. (S. Bemerkung (2) bei (3. 3).)

IV. VERALLGEMEINERTE FAKTOREN

§ 14. Allgemeine Existenzsätze

(14. 1) Es sei Γ ein nichtleerer Graph, $\varkappa(X)$ und $\varkappa'(X)$ seien auf der Menge der Punkte von Γ definierte Funktionen, die nur nichtnegative ganze Werte annehmen. Ein zu $\varkappa$ und $\varkappa'$ gehöriger *verallgemeinerter Faktor* von Γ oder kurz ein *$(\varkappa, \varkappa')$-Faktor* ist ein solches Bogensystem H von Γ, das für jeden Punkt X von Γ den Bedingungen

(1) $$[H, X] = \varkappa(X) \quad \text{und} \quad |H, X| \leqq \varkappa'(X)$$

genügt. Nach (4. 2) können wir auch sagen: Ein $(\varkappa, \varkappa')$-Faktor ist ein aufnehmbares Bogensystem H mit $\delta(H)=0$. Ist für jedes X $\varkappa'(X)=0$, so sind die $(\varkappa, \varkappa')$-Faktoren mit den gewöhnlichen $\varkappa$-Faktoren identisch (s. [10] und [7]).

Sind $\varkappa$ und $\varkappa'$ zueinander komplementär (s. (12. 1)), so sollen die $(\varkappa, \varkappa')$-Faktoren *topologische $\varkappa$-Faktoren* heißen.

Sind endlich zu dem Graphen Γ keine Funktionen $\varkappa$ und $\varkappa'$ angegeben, sondern sind nur in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet, so verstehen wir unter einem *topologischen σ-Faktor* — wo σ eine natürliche Zahl bedeutet — einen topologischen $\varkappa$-Faktor, bei dem die Funktion $\varkappa(X)$ in jedem Hauptpunkt den Wert σ, in jedem Nebenpunkt den Wert 0 annimmt.

Aus unserer Erklärung und aus (4. 2) folgt unmittelbar die Behauptung: Γ besitzt dann und nur dann einen $(\varkappa, \varkappa')$-Faktor, wenn

$$\nu_{\max}=\frac{1}{2}\varkappa(\Phi) \tag{2}$$

besteht. Daraus beweisen wir den

Satz (14. 2) *Es seien in den Punkten von Γ die Kapazitätsfunktionen $\varkappa(X)$ und $\varkappa'(X)$ erklärt. Γ enthält dann und nur dann einen $(\varkappa, \varkappa')$-Faktor, wenn für jedes Gewichtssystem $q=q(A, B, C)$*

$$\frac{1}{2}\varkappa(\Phi)\leqq S(q) \tag{1}$$

besteht, oder anders formuliert, wenn bei jeder Zerlegung der Menge Φ in drei Mengen A, B und C die Ungleichung

$$\varkappa(A)\leqq\varkappa(B)+2\varkappa'(B)+2\nu(A, A)+\nu(A, C)-\tau \tag{2}$$

besteht, wo τ die Anzahl der Komponenten $[C_i]$ von $[C]$ mit $\varkappa(C_i)+\nu(A, C_i)\equiv 1$ (mod 2) bedeutet.

Beweis. Nach (4. 1) ist $\nu_{\max}\leqq S(q)$. Besteht also (14. 1) (2), so gilt für jedes q die Ungleichung (1). Gilt umgekehrt für jedes q die Ungleichung (1), so besteht laut unseres Hauptsatzes (3. 3) die Ungleichung $\nu_{\max}\geqq\frac{1}{2}\varkappa(\Phi)$, woraus nach (4. 2) (2) die Gleichung (14. 1) (2) folgt.

Die Ungleichung (1) ist unter Beachtung von $\varkappa(\Phi)=\varkappa(A)+\varkappa(B)+\varkappa(C)$ mit (2) gleichwertig.

Bemerkung. Ist für jedes X $\varkappa'(X)=0$, so gibt (14. 2) die Belck—Tuttesche Bedingung der Existenz von σ- bzw. $\varkappa$-Faktoren endlicher Graphen an ([2], Theorem IV; [10], Theorem XV).

Wir wollen uns mit den topologischen 1- und 2-Faktoren auch separiert beschäftigen.

(14. 3) Betrachten wir zuerst die topologischen 1-Faktoren. Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Die Definitionen von $\varkappa$ und $\varkappa'$ lauten dann: Es ist in jedem Hauptpunkt $\varkappa(X)=1$ und $\varkappa'(X)=0$, in jedem Nebenpunkt $\varkappa(X)=0$ und $\varkappa'(X)=1$. Nach (11. 5) und (11. 6) genügt es jetzt, in (14. 2) nur Systeme $q=q(A, B, C)$ mit $A=\emptyset$ zu betrachten.

Wir bekommen so aus (14. 2) (2) mit den Bezeichnungen von (12. 2) den

Satz (14. 4) *Es seien irgendwelche Punkte in Γ als Hauptpunkte ausgezeichnet. Γ enthält dann und nur dann einen topologischen 1-Faktor, d. h. $\frac{1}{2}\nu(\Phi)$ unabhängige Hauptwege, wenn für jede beliebige Teilmenge B von Φ die Ungleichung*

(1) $$\tau(\bar{B}) \leqq \nu_h(B)+2\nu_n(B)$$

besteht.

Sind sämtliche Punkte von Γ Hauptpunkte, so gibt (14. 4) den bekannten Tutteschen Satz über die Existenz von 1-Faktoren an ([9], Theorem IV).

(14. 5) Jetzt wollen wir uns mit topologischen 2-Faktoren beschäftigen. Es seien daher wieder irgendwelche Punkte in Γ als Hauptpunkte ausgezeichnet, und $\varkappa$ und $\varkappa'$ sollen folgendermaßen definiert sein: In jedem Hauptpunkt ist $\varkappa(X)=2$ und $\varkappa'(X)=0$, in jedem Nebenpunkt $\varkappa(X)=0$ und $\varkappa'(X)=1$.

Existieren Wege in einem topologischen 2-Faktor, so setzen sich diese zu Kreisen zusammen. Diese Kreise bilden, zusammen mit den Schlingen des Faktors, ein *Kreissystem* mit folgenden Eigenschaften: Jeder Kreis des Systems enthält mindestens einen Hauptpunkt, jeder Hauptpunkt von Γ liegt auf einem Kreis des Systems und es haben je zwei Kreise des Systems (falls mehrere existieren) keinen gemeinsamen Punkt. Auf diese Weise gehört zu jedem topologischen 2-Faktor ein Kreissystem. (Zu dem leeren Faktor gehört das leere Kreissystem.) Umgekehrt ist es klar, daß die Hauptpunkte von Γ jedes Kreissystem mit den erwähnten Eigenschaften in solche Bogen zerlegen, die zusammen einen topologischen 2-Faktor bilden.

Nach (11. 5) und (11. 7) genügt es bei topologischen 2-Faktoren in (14. 2) nur Systeme q mit den Eigenschaften (11. 7) (1), (2) zu betrachten, und so erhält man aus (14. 2) (2) den

Satz (14. 6) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Γ besitzt dann und nur dann einen topologischen 2-Faktor, wenn für jede solche Zerlegung von Φ in drei Teilmengen A, B und C, bei welcher*

A aus unabhängigen Hauptpunkten besteht, und zu jedem C-Punkt höchstens eine AC-Kante inzident ist, die Ungleichung

$$\nu(A) \leqq \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2} \nu(A, C_i) \right] = \nu(B) + \frac{1}{2} (\nu(A, C) - \tau)$$

gültig ist. Hier bezeichnen $[C_i]$ $(i = 1, \ldots, m)$ *die Komponenten von* $[C]$ *und* τ *die Anzahl der Komponenten* $[C_i]$ *mit ungeradem* $\nu(A, C_i)$.

Wir wollen aus (14. 6) zwei weitere Bedingungen der Existenz topologischer 2-Faktoren ableiten.

(14. 7) Es sei A eine beliebige Menge *unabhängiger* Punkte. Wir wollen jeden Bogen, dessen beide Randpunkte A-Punkte sind, die inneren Punkte jedoch nicht, *A-Bogen* nennen. Wir nennen das Bogensystem H ein *System unabhängiger A-Bogen*, wenn jeder Bogen von H (falls $H \neq \emptyset$ ist) ein A-Bogen ist und je zwei Bogen von H keinen gemeinsamen *inneren* Punkt enthalten. Ein Gewichtssystem r zweiter Art (s. (12. 5)) nennen wir ein *zu A gehöriges Gewichtssystem*, wenn in r alle A-Punkte, falls $A \neq \emptyset$, das Gewicht 0 erhalten und jeder A-Bogen, falls solche existieren, einen Punkt mit dem Gewicht 1 oder zwei Punkte mit halben Gewichten enthält. Es ist klar, daß zu A gehörige Systeme existieren. Wir wollen für ein jedes zu A gehöriges r einen *zu A gehörigen Wert* $S_A(r)$ definieren. Bezeichnet wieder B die Menge der Punkte, die in r das Gewicht 1 erhalten, und $\nu_{1/2}(E)$ die Anzahl der Punkte von E $(E \subseteq \Phi)$, die in r das Gewicht 1/2 bekommen, so sei

$$S_A(r) = \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2} \nu_{1/2}(C_i) \right],$$

wo $[C_i]$ $(i = 1, \ldots, m)$ die Komponenten von $[C]$ mit $C = \Phi - (A \cup B)$ bedeuten.

Durch die selbe Schlußweise, die im Teile (II) des Beweises von (12. 6) benutzt wurde, kann man die Richtigkeit folgender Behauptung einsehen:

(14. 8) *Es sei A eine beliebige Menge unabhängiger Punkte von* Γ *und r ein zu A gehöriges Gewichtssystem.* Ist H ein System *unabhängiger A-Bogen, so gilt* $\nu(H) \leqq S_A(r)$.

Wir beweisen folgenden

Satz (14. 9) *Es seien in* Γ *irgendwelche Punkte als Hauptpunkte ausgezeichnet.* Γ *enthält dann und nur dann einen topologischen 2-Faktor, wenn für jede beliebige Menge A unabhängiger Hauptpunkte und für jedes zu A gehöriges Gewichtssystem r die Ungleichung* $\nu(A) \leqq S_A(r)$ *besteht.*

Beweis. (I) Wir nehmen an, daß Γ einen topologischen 2-Faktor H besitzt. Es sei ferner A eine beliebige Menge unabhängiger Hauptpunkte und r ein beliebiges zu A gehöriges Gewichtssystem. Jene Kreise des zu H gehörigen

Kreissystems (s. (14. 5)), die A-Punkte enthalten, werden durch diese Punkte in genau $\nu(A)$ unabhängige A-Bogen zerlegt. Nach (14. 8) ist dann $\nu(A) \leqq S_A(r)$.

(II) Besitzt Γ keinen topologischen 2-Faktor, so kann man nach (14. 6) Φ so in die Teilmengen A, B und C zerlegen, daß A aus unabhängigen Hauptpunkten bestehe, zu jedem C-Punkt höchstens eine AC-Kante inzident sei und

$$(1) \qquad \nu(A) > \nu(B) + \sum_{i=1}^{m} \left[\frac{1}{2} \nu(A, C_i) \right]$$

gelte. ($[C_i]$ $(i=1, \ldots, m)$ sind die Komponenten von $[C]$.) Aus (1) ergibt sich $A \neq \emptyset$. Wir definieren nun ein Gewichtssystem r wie folgt: Erhalte in r jeder B-Punkt das Gewicht 1, jeder C-Punkt, zu dem eine AC-Kante inzident ist, das Gewicht 1/2, und alle anderen Punkte das Gewicht 0. Dann ist es klar, daß r ein zu A gehöriges System ist und $S_A(r)$ der rechten Seite von (1) gleich ist. Damit haben wir unseren Satz bewiesen.

Es gilt der

Satz (14. 10) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Γ enthält dann und nur dann einen topologischen 2-Faktor, wenn zu jeder Menge A unabhängiger Hauptpunkte ein Bogensystem aus mindestens $\nu(A)$ unabhängigen A-Bogen existiert.*

Beweis. Die Notwendigkeit unserer Bedingung ist trivial. Nehmen wir nun an, daß kein topologischer 2-Faktor existiert. Dann gibt es nach (14. 9) eine Menge A unabhängiger Hauptpunkte und ein zu A gehöriges Gewichtssystem r mit $\nu(A) > S_A(r)$. Ist H ein beliebiges System unabhängiger A-Bogen, so besteht nach (14. 8) $\nu(H) \leqq S_A(r)$. Es gilt also $\nu(H) < \nu(A)$.

Bemerkung. Ist jeder Punkt in Γ Hauptpunkt, so geben die Sätze (14. 6), (14. 9) und (14. 10) Bedingungen der Existenz gewöhnlicher 2-Faktoren an. Die aus (14. 9) sich ergebende Bedingung ist einem Tutteschen Ergebnis über gerichtete Graphen ähnlich ([11], Satz (5. 1)).

§ 15. Topologische 1-Faktoren bei speziellen Graphen

In diesem Paragraphen wollen wir aus dem Satze (14. 4) für gewisse spezielle Graphen einige hinreichende Bedingungen der Existenz topologischer 1-Faktoren ableiten.

(15. 1) Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Wir wollen zu Γ eine Größe ξ' definieren, die eine Zusammenhangseigenschaft von Γ bezüglich der Hauptpunkte ausdrückt. Es bezeichne Δ die Menge derjenigen Teilmengen von Φ, die eine ungerade Anzahl von Haupt-

punkten enthalten. Gibt es in Φ mindestens einen Hauptpunkt, so ist Δ nichtleer, und in diesem Falle sei

$$\xi' = \min_{E \in \Delta} r(E, \bar{E}) \qquad (\bar{E} = \Phi - E).$$

Enthält Φ keinen Hauptpunkt, so sei $\xi' = \infty$.

In den nachfolgenden Sätzen werden einige Bedingungen mehrmals vorkommen. Der Kürze halber wollen wir diese von den Sätzen getrennt formulieren.

Bedingung (a). Der Grad eines jeden Hauptpunktes ist gleich μ $(\mu \geqq 1)$, eines jeden Nebenpunktes kleiner oder gleich 2μ.

Bedingung (b). Jede Komponente von Γ enthält eine gerade Anzahl von Hauptpunkten.

Bedingung (c). Alle Nebenpunkte sind — falls solche existieren — geraden Grades.

Es gilt folgender Satz:

Satz (15. 2) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Bestehen dann die Bedingungen* (a), (b) *und* $\xi' \geqq \mu$, *so besitzt Γ einen topologischen* 1-*Faktor.*

Beweis. Es genüge Γ den angeführten Bedingungen und sei B eine beliebige Teilmenge von Φ. Wir wollen bezüglich B die Bezeichnungen unter (12. 2) anwenden. Ist B leer, so ist nach (b) $r(\bar{B}) = 0$, und so gilt (14. 4) (1). Es sei nun B nichtleer. Ist die Komponente $[B_i]$ ungerade, so ist $\bar{B}_i \in \Delta$, und demzufolge ist $r(B_i, B) \geqq \xi' \geqq \mu$. Daraus folgt $r(B, B) \geqq \mu\, r(B)$. (Dies ist auch im Falle $\xi' = \infty$ richtig!) Nach (a) ist aber $r(\bar{B}, B) \geqq \mu\, r_{\mathrm{I}}(B) + 2\mu\, r_{\mathrm{II}}(B)$ und wir können daher feststellen, daß B der Ungleichung (14. 4) (1) genügt. Nach (14. 4) enthält also Γ einen topologischen 1-Faktor.

Bemerkung. Im Falle $\mu = 1$ folgt das Bestehen von $\xi' \geqq \mu$ aus (b).

Genügt Γ neben (a) und (b) auch noch der Bedingung (c), so kann man statt $\xi' \geqq \mu$ eine schwächere Forderung stellen:

Satz (15. 3) *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Bestehen dann die Bedingungen* (a), (b), (c) *und* $\xi' \geqq \mu - 1$, *so besitzt Γ einen topologischen* 1-*Faktor.*

Beweis. Ist jetzt $E \in \Delta$, so ist ($\varrho(X)$ Bezeichnet den Grad von X)

$$r(E, \bar{E}) = \sum_{X \in E} \varrho(X) - 2\, r(E, E) \equiv \mu\, r_{\mathrm{I}}(E) \equiv \mu \pmod 2.$$

Ist also ξ' endlich, so gilt $\xi' \equiv \mu$ (mod 2), und dann folgt aus dem Bestehen

von $\xi' \geqq \mu-1$ auch $\xi' \geqq \mu$. Die letzte Behauptung ist auch im Falle $\xi' = \infty$ richtig. Dann folgt aber aus (15. 2) die Behauptung von (15. 3).

Bemerkung. Im Falle $\mu = 2$ folgt das Bestehen von $\xi' \geqq \mu-1$ aus (b).

(15. 4) Für die Werte $\mu = 1, 2, 3$ und 4 wollen wir in den Sätzen (15. 2) und (15. 3) die Größe ξ' durch anschaulichere Begriffe ersetzen.

Vermehren sich die Komponenten des Graphen Γ durch Weglassen der Kante x bzw. der Kanten x_1 und x_2, so heißt x eine *Brücke* bzw. das Kantenpaar (x_1, x_2) eine *Doppelbrücke* (von Γ). Wir formulieren mit diesen Begriffen zwei weitere Bedingungen.

Bedingung (d). Γ enthält keine Brücke.

Bedingung (e). Γ enthält keine Doppelbrücke.

Die folgenden Behauptungen sind leicht ersichtlich:

Aus (b) und (d) folgt $\xi' \geqq 2$.

Aus (b) und (e) folgt $\xi' \geqq 3$, falls Γ mindestens zwei Kanten enthält.

Wir können demnach aus (15. 2) und (15. 3) zum folgenden Satz gelangen:

Satz (15. 5). *Es seien in Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet. Das Bestehen der folgenden Bedingungen sichert dann die Existenz eines topologischen* 1*-Faktors von Γ:*

Im Falle $\mu = 1$ die Bedingungen (a), (b);
im Falle $\mu = 2$ die Bedingungen (a), (b), (d) *oder* (a), (b), (c);
im Falle $\mu = 3$ die Bedingungen (a), (b), (e) *oder* (a), (b), (c), (d);
im Falle $\mu = 4$ die Bedingungen (a), (b), (c), (e).

Bemerkungen. (1) Es existieren Graphen, die der Bedingung (a) mit $\mu = 3$, sowie den Bedingungen (b) und (d) genügen, und die keinen topologischen 1-Faktor enthalten.

(2) Sind sämtliche Punkte Hauptpunkte, so geben (15. 2), (15. 3) und (15. 5) bekannte Sätze über 1-Faktoren an (s. [2], [3], [4], [9]).

§ 16. Topologische $\varkappa$-Faktoren bei speziellen Graphen

In diesem Paragraphen leiten wir aus (14. 2) einige hinreichende Bedingungen der Existenz topologischer $\varkappa$-Faktoren her (s. [2], S. 247 und [4], S. 144—146).

(16. 1) Es seien in den Punkten von Γ die Kapazitätsfunktionen $\varkappa(X)$ und $\varkappa'(X)$ vorläufig beliebig definiert.

Wir nehmen nun an, daß Γ *keinen* $(\varkappa, \varkappa')$*-Faktor besitzt* und wollen aus dieser Annahme, vorausgesetzt, daß Γ, $\varkappa$ und $\varkappa'$ gewisse Bedingungen erfül-

len, eine Ungleichung (die Ungleichung (8)) ableiten. Dann werden wir solche Forderungen stellen, die dieser Ungleichung widersprechen. Diese Forderungen, zusammen mit den vorher erwähnten Bedingungen, werden dann die Existenz gewisser topologischer Faktoren sichern.

BEDINGUNG (a). Γ ist zusammenhängend.

BEDINGUNG (b) Die Zahl $\varkappa(\Phi)$ ist gerade.

Wir nehmen an, daß Γ den Bedingungen (a) und (b) genügt.

Da Γ keinen $(\varkappa, \varkappa')$-Faktor enthält, kann man nach (14. 2) die Menge Φ so in drei Teilmengen A, B und C zerlegen, daß

$$\varkappa(A) > \varkappa(B) + 2\varkappa'(B) + 2\nu(A, A) + \nu(A, C) - \tau \tag{1}$$

besteht, wo τ die Anzahl der ungeraden Komponenten von $[C]$ bedeutet. (Die $[C_i]$ $(i = 1, \ldots, m)$ bezeichnen die Komponenten von $[C]$, und ein $[C_i]$ heißt ungerade, wenn $\varkappa(C_i) + \nu(A, C_i)$ ungerade ist.)

Es ist $A \cup B \neq \emptyset$. Ist nämlich $A = B = \emptyset$, so gilt $C = \Phi$, und so nach (a) und (b) auch $\tau = 0$, was zu (1) in Widerspruch steht. Aus $A \cup B \neq \emptyset$ und (a) folgt, daß zu jedem nichtleeren $[C_i]$ entweder eine AC_i-Kante oder eine BC_i-Kante existiert.

Wir nennen ein ungerades $[C_i]$ eine C_A- bzw. C_B-*Komponente*, wenn die Kanten, die den Teilgraphen $[C_i]$ berühren, alle AC_i- bzw. BC_i-Kanten sind. Es bezeichne τ_a bzw. τ_b die Anzahl der C_A- bzw. C_B-Komponenten, und J bedeute die Menge jener Indizes i, zu denen solche $[C_i]$ gehören, die C_A- oder C_B-Komponenten sind. Ist J nichtleer, so sei

$$\xi_\tau = \min_{i \in J} \nu(C_i, \bar{C}_i) \qquad (\bar{C}_i = \Phi - C_i).$$

Ist J leer, so sei $\xi_\tau = \nu(\Psi) + 3$. (Ψ bedeutet die Menge der Kanten von Γ.) In jedem Falle ist $\xi_\tau \geqq 1$ und es laufen aus jeder C_A- bzw. C_B-Komponente nach A bzw. B mindestens ξ_τ Kanten. Ferner gilt: Abgesehen von den C_B- bzw. C_A-Komponenten läuft aus jedem ungeraden $[C_i]$ mindestens eine Kante nach A bzw. nach B. Wir können nun folgende Ungleichungen feststellen:

$$\nu(A, C) \geqq \tau_a \xi_\tau + \tau - \tau_a - \tau_b, \tag{2}$$

$$\nu(B, C) \geqq \tau_b \xi_\tau + \tau - \tau_a - \tau_b. \tag{3}$$

Aus (1) und (2) folgt

$$(\xi_\tau - 1)\tau_a - \tau_b + 2\varkappa'(B) < \varkappa(A) - \varkappa(B). \tag{4}$$

Für die Zahl $\nu(B, B)$ gilt offensichtlich ($\varrho(X)$ ist der Grad von X)

$$\nu(B, \bar{B}) = \varrho(B) - 2\nu(B, B) \leqq \varrho(B). \qquad (\bar{B} = \Phi - B) \tag{5}$$

Wir wollen $\nu(B, \bar{B})$ auch von unten abschätzen. Es gilt

$$\nu(A, B) = \varrho(A) - 2\nu(A, A) - \nu(A, C).$$

Es folgt daraus unter Beachtung von $\nu(B, \bar{B}) = \nu(B, A) + \nu(B, C)$ sowie von (3) und (1)

$$\nu(B, \bar{B}) > \varrho(A) - \varkappa(A) + \varkappa(B) + 2\varkappa'(B) - \tau_a + (\xi_\tau - 1)\tau_b.$$

Dies ergibt zusammen mit (5) die Ungleichung

$$-\tau_a + (\xi_\tau - 1)\tau_b + 2\varkappa'(B) < \varrho(B) - \varrho(A) + \varkappa(A) - \varkappa(B). \tag{6}$$

Von hier an wollen wir uns auf topologische $\varkappa$-Faktoren beschränken, also nehmen an: $\varkappa$ und $\varkappa'$ sind komplementäre Funktionen (s. (12. 1)). Ferner sollen Γ und $\varkappa(X)$ noch der folgenden Bedingung genügen (nach (12. 1) können wir die Bezeichnungen Haupt- und Nebenpunkte benützen):

Bedingung (c). Es existiert eine Zahl λ mit $0 < \lambda < 1$, so daß für jeden Hauptpunkt $\varkappa(X) = \lambda\varrho(X)$ und für jeden Nebenpunkt $\varrho(X) \leqq 2/\lambda$ besteht.

Es folgt aus (c) (da $\varkappa$ und $\varkappa'$ komplementär sind) für jeden Punkt X von Γ die Ungleichung

$$\varkappa(X) \leqq \lambda\varrho(X) \leqq \varkappa(X) + 2\varkappa'(X).$$

Daraus ergibt sich

$$\varkappa(A) - \varkappa(B) \leqq \lambda(\varrho(A) - \varrho(B)) + 2\varkappa'(B). \tag{7}$$

(4) und (7) bzw. (6) und (7) ergeben die Ungleichungen

$$(\xi_\tau - 1)\tau_a - \tau_b < \lambda(\varrho(A) - \varrho(B)),$$

$$-\tau_a + (\xi_\tau - 1)\tau_b < (1 - \lambda)(\varrho(B) - \varrho(A)).$$

Aus diesen erhalten wir endlich die gewünschte Ungleichung

$$((1 - \lambda)(\xi_\tau - 1) - \lambda)\tau_a + (\lambda(\xi_\tau - 1) - (1 - \lambda))\tau_b < 0. \tag{8}$$

Fordern wir nun, daß in (8) die Koeffizienten von τ_a und τ_b nichtnegativ seien, d. h. daß die beiden Ungleichungen

$$\xi_\tau \geqq 1/\lambda \quad \text{und} \quad \xi_\tau \geqq 1/(1 - \lambda) \tag{9}$$

bestehen, dann haben wir einen Widerspruch.

Wir definieren die Zahl ξ, die eine Zusammenhangseigenschaft von Γ charakterisiert, folgendermaßen: Ist $\nu(\Phi) > 1$, so sei

$$\xi = \min_{\emptyset \subset E \subset \Phi} \nu(E, \bar{E}) \qquad (\bar{E} = \Phi - E).$$

Ist $\nu(\Phi) = 1$, so sei $\xi = 0$.

Da in jedem Falle $\xi \leqq \nu(\Psi)$ ist, gilt offensichtlich $\xi_\tau \geqq \xi$.

Wir formulieren die

Bedingung (d). Es gilt

$$\xi \geqq \max\big(1/\lambda,\ 1/(1-\lambda)\big).$$

Nach den obigen können wir dann folgenden Satz aussprechen:

Satz (16. 2) *Genügen Γ und $\varkappa$ den Bedingungen* (a), (b), (c) *und* (d), *so enthält Γ einen topologischen $\varkappa$-Faktor.*

Bemerkung. Sind sämtliche Punkte Hauptpunkte, so bekommt man aus (16. 2) eine schwächere Form eines Oreschen Satzes über $\varkappa$-Faktoren ([7], Theorem 3. 2. 1).

Wir wollen den wichtigsten Spezialfall von (16. 2) auch separiert formulieren:

Satz (16. 3) *Es seien im zusammenhängenden Graphen Γ irgendwelche Punkte als Hauptpunkte ausgezeichnet und seien μ und $\sigma = \lambda\mu$ $(0<\lambda<1)$ natürliche Zahlen sowie $\sigma\nu_h(\Phi)$ eine gerade Zahl. Gilt für jeden Hauptpunkt $\varrho(X)=\mu$, für jeden Nebenpunkt $\varrho(X)\leqq 2\mu/\sigma$ und genügt Γ der Bedingung* (d), *so enthält Γ einen topologischen σ-Faktor.*

(16. 4) Nehmen wir jetzt an, daß die in (16. 1) gestellten Annahmen gelten, und daß $\varkappa(X)$ neben (a) und (c) noch der folgenden Bedingung genügt:

Bedingung (e). $\varkappa(X)$ ist für jedes X gerade.

Es können dann C_B-Komponenten gar nicht existieren, also ist $\tau_b = 0$. Ferner gilt für jede beliebige C_a-Komponente $[C_i]$

$$\nu(C_i, \bar{C}_i) \equiv \nu(C_i, A) \equiv \varkappa(C_i) + \nu(C_i, A) \equiv 1 \pmod 2.$$

Enthält Γ keine Brücke, so folgt daraus $\xi_r \geqq 3$, also steht die Forderung $1/(1-\lambda) \leqq 3$ zu (16. 1) (8) in Widerspruch. Es gilt daher der

Satz (16. 5) *Genügen Γ und $\varkappa$ den Bedingungen* (a), (c), (e) *und enthält Γ keine Brücke, so besitzt Γ im Falle $\lambda \leqq 2/3$ einen topologischen $\varkappa$-Faktor.*

Wir wollen diesen Satz für topologische 2-Faktoren auch separiert formulieren.

Satz (16. 6) *Es seien im zusammenhängenden Graphen irgendwelche Punkte als Hauptpunkte ausgezeichnet und sei die natürliche Zahl $\mu \geqq 3$. Gilt für jeden Hauptpunkt $\varrho(X)=\mu$, für jeden Nebenpunkt $\varrho(X)\leqq\mu$, und enthält Γ keine Brücke, so besitzt Γ einen topologischen 2-Faktor, oder — was damit gleichwertig ist — Γ enthält ein solches Kreissystem, dessen Kreise (falls mehrere existieren) paarweise keinen gemeinsamen Punkt haben und zusammen alle Hauptpunkte enthalten.*

Sind sämtliche Punkte Hauptpunkte, so ergibt dieser Satz bekannte Sätze über 2-Faktoren (s. [1], [2], [4], [8]).

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SUR UNE GÉNÉRALISATION DU THÉORÈME DE POULAIN ET HERMITE POUR LES ZÉROS RÉELS DES POLYNÔMES RÉELS

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Soit

(1) $$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

un polynôme dont tous les zéros sont réels. D'après le théorème classique de POULAIN—HERMITE, si le polynôme

(2) $$g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m$$

a seulement des zéros réels, le polynôme

(3) $$g(D) f(x) = b_0 f^{(m)}(x) + b_1 f^{(m-1)}(x) + \cdots + b_m f(x)$$

a aussi seulement des zéros réels. Chaque zéro multiple de (3) est aussi un zéro multiple de (1). Dans ce travail nous démontrons la généralisation suivante de ce théorème:

1. *Supposons que le polynôme* (1) *a seulement des zéros réels et que pour les arguments* φ *des zéros imaginaires du polynôme* (2), *dont tous les coefficients sont réels, on a l'inégalité*

(4) $$|\sin \varphi| \leqq \frac{1}{\sqrt{n}}.$$

Alors le polynôme (3) *a seulement des zéros réels. Si dans* (4) *on a le signe d'inégalité, chaque zéro multiple de* (3) *est aussi un zéro multiple de* (1).

La démonstration est basée sur la proposition nouvelle:

2. *Supposons que le polynôme* (1) *a seulement des zéros réels et soit* φ *un angle satisfaisant à la condition* (4). *Alors le polynôme*

(5) $$F(x) = f(x) - 2\varrho \cos \varphi f'(x) + \varrho^2 f''(x) \qquad (\varrho > 0)$$

a tous ses zéros réels et, si dans (4) *on a le signe d'inégalité, chaque zéro multiple de* (5) *est en même temps un zéro multiple de* (1).

Nous considérons d'abord le cas où (1) n'a pas des zéros multiples. D'après le théorème de POULAIN—HERMITE pour $\varphi = 0$ le polynôme (5) aura seulement des zéros réels et simples. Puisque les zéros de (5) sont des fonctions continues de φ pour $\varphi > 0$ assez petit, le polynôme (5) aura seulement

des zéros réels et simples. Lorsque φ croît, aucun zéro réel de (5) ne peut devenir imaginaire sans se confondre avec un autre zéro. Désignons alors par φ_0 la plus petite valeur de φ, pour laquelle (5) a au moins un zéro multiple. Nous démontrerons que

$$\cos^2\varphi_0 < 1 - \frac{1}{n}. \tag{6}$$

Désignons par λ un zéro multiple de (5) $(\varphi = \varphi_0)$ et posons $\alpha = -\varrho\cos\varphi$. De

$$f(x) = c_0 + c_1(x-\lambda) + c_2(x-\lambda)^2 + \cdots + c_n(x-\lambda)^n$$

on obtient

$$F(x) = D_0 + D_1(x-\lambda) + D_2(x-\lambda)^2 + \cdots,$$

où

$$D_0 = c_0 + 2\alpha c_1 + 2\varrho^2 c_2, \quad D_1 = c_1 + 4\alpha c_2 + 6\varrho^2 c_3.$$

Comme λ est un zéro multiple de (5) il faut avoir

$$c_0 + 2\alpha c_1 + 2\varrho^2 c_2 = 0, \quad c_1 + 4\alpha c_2 + 6\varrho^2 c_3 = 0. \tag{7}$$

Soit $c_0 = 0$. Alors il suit que $c_1 \neq 0$, sinon (1) aura des zéros multiples. Pour $n = 2$ les équations (7) sont les suivantes: $\alpha c_1 + \varrho^2 c_2 = 0$, $c_1 + 4\alpha c_2 = 0$, d'où l'on obtient que $4\alpha^2 = \varrho^2$, c'est-à-dire $\cos^2\varphi_0 = \frac{1}{4} < \frac{1}{2}$. Pour $n \geqq 3$, on obtient de (7)

$$\frac{\alpha^2}{\varrho^2} = \cos^2\varphi_0 = \frac{c_2^2}{2(2c_2^2 - 3c_1c_3)}.$$

Nous allons démontrer que

$$\frac{c_2^2}{2(2c_2^2 - 3c_1c_3)} < 1 - \frac{1}{n}. \tag{8}$$

Le polynôme

$$c_1 + \binom{n-1}{1}\frac{c_2}{\binom{n-1}{1}}x + \binom{n-1}{2}\frac{c_3}{\binom{n-1}{2}}x^2 + \cdots$$

a tous ses zéros réels. D'après une inégalité connue d'Euler, on aura

$$\left(\frac{c_2}{n-1}\right)^2 \geqq \frac{c_1c_3}{\binom{n-1}{2}}. \tag{9}$$

De cette inégalité, il suit d'abord que $c_2^2 > \frac{3}{2}c_1c_3$. Donc dans (8) la division est admissible. Ensuite de (9) il découle facilement l'inégalité (8).

Considérons maintenant le cas où $c_0 \neq 0$. Des équations (7), on a

$$\frac{\alpha^2}{\varrho^2} = \cos^2 \varphi_0 = \frac{(3c_0c_3 - c_1c_2)^2}{2(c_1^2 - 2c_0c_2)(2c_2^2 - 3c_1c_3)}.$$

Il est bien connu que le nombre $c_1^2 - 2c_0c_2$ est positif. Le nombre $2c_2^2 - 3c_1c_3$ est aussi positif. En effet, pour $n = 2$ c'est évident. Pour $n \geqq 3$ l'inégalité $2c_2^2 - 3c_1c_3 > 0$ découle de l'inégalité d'Euler

$$2c_2^2 \geqq 3\frac{n-1}{n-2}c_1c_3$$

concernant le polynôme $c_0 + c_1x + c_2x^2 + \cdots$. Nous démontrerons l'inégalité générale

$$\frac{(3c_0c_3 - c_1c_2)^2}{2(c_1^2 - 2c_0c_2)(2c_2^2 - 3c_1c_3)} < 1 - \frac{1}{n}. \tag{10}$$

Pour $n = 2$ $(c_3 = 0)$ cette inégalité devient

$$\frac{c_1^2}{2(c_1^2 - 2c_0c_2)} < 1,$$

ce qui est équivalent à l'inégalité connue $c_1^2 > 4c_0c_2$. Donc on peut supposer que $n \geqq 3$. D'après les conditions on a

$$c_0 + c_1x + \cdots + c_nx^n = c_0(1 + x_1x)(1 + x_2x)\cdots(1 + x_nx), \tag{11}$$

où les nombres $x_1, x_2, \ldots, x_n$ sont réels. De (11) on obtient que

$$\frac{c_1}{c_0} = \Sigma x_1, \quad \frac{c_2}{c_0} = \Sigma x_1x_2, \quad \frac{c_3}{c_0} = \Sigma x_1x_2x_3,$$

d'où l'on a

$$\frac{1}{c_0^2}(c_1^2 - 2c_0c_2) = \Sigma x_1^2,$$

$$\frac{1}{c_0^2}(c_1c_2 - 3c_0c_3) = \Sigma x_1^2x_2,$$

$$\frac{1}{c_0^2}(2c_2^2 - 3c_1c_3) = 2(\Sigma x_1^2x_2^2 + 2\Sigma x_1^2x_2x_3 + 6\Sigma x_1x_2x_3x_4),$$
$$-3(\Sigma x_1^2x_2x_3 + 4\Sigma x_1x_2x_3x_4) = 2\Sigma x_1^2x_2^2 + \Sigma x_1^2x_2x_3.$$

Donc l'inégalité (10) prend la forme

$$\frac{(\Sigma x_1^2x_2)^2}{2\Sigma x_1^2(2\Sigma x_1^2x_2^2 + \Sigma x_1^2x_2x_3)} < 1 - \frac{1}{n},$$

où

$$U = 2(n-1)\Sigma x_1^2(2\Sigma x_1^2x_2^2 + \Sigma x_1^2x_2x_3) - n(\Sigma x_1^2x_2)^2 > 0. \tag{12}$$

Nous démontrerons maintenant l'identité suivante:

$$U = \sum_{i<j}^{1\ldots n} L_{ij}(x_i - x_j)^2, \tag{13}$$

où L_{ij} sont les nombres

$$L_{ij} = \left(\sum_{s=1}^{n} x_s^2 - x_i^2 - x_j^2\right) \sum_{s=1}^{n} x_s^2 + n x_i^2 x_j^2.$$

En effet, pour la forme $(\Sigma x_1^2 x_2)^2$ nous avons

$$(\Sigma x_1^2 x_2)^2 = \Sigma x_1^4 x_2^2 + 2\Sigma x_1^3 x_2^3 + 2\Sigma x_1^3 x_2^2 x_3 + 4\Sigma x_1^2 x_2^2 x_3 x_4 + 6\Sigma x_1^2 x_2^2 x_3^2 + 2\Sigma x_1^4 x_2 x_3.$$

Désignons l'expression $\sum_{s=1}^{n} x_s^2$ par V. Nous allons trouver dans (12) tous les termes qui contiennent le produit $x_1 x_2$, dont les multiplicateurs sont du degré pair relativement les variables $x_1, x_2, \ldots, x_n$. Il est évident que nous nous devrons borner à la partie

$$2(n-1)\Sigma x_1^2 \Sigma x_1^2 x_2 x_3 - n(\Sigma x_1^2 x_2)^2$$

de la forme U. On voit facilement que le multiplicateur cherché de $x_1 x_2$ dans U est égal à

$$2(n-1)V(V - x_1^2 - x_2^2) - n\left[2x_1^2 x_2^2 + 2(x_1^2 + x_2^2)\sum_{p=3}^{n} x_p^2\right] - 4n\sum_{i<j}^{3\ldots n} x_i^2 x_j^2 - 2n\sum_{p=3}^{n} x_p^2 =$$

$$= 2n(V - x_1^2 - x_2^2)^2 - 2V(V - x_1^2 - x_2^2) - 4n\sum_{i<j}^{3\ldots n} x_i^2 x_j^2 - 2n\sum_{p=3}^{n} x_p^4 - 2n x_1^2 x_2^2 =$$

$$= -2V(V - x_1^2 - x_2^2) - 2n x_1^2 x_2^2.$$

Donc en désignant par L_{ij} le coefficient de $-2x_1 x_2$ on aura

$$L_{12} = V(V - x_1^2 - x_2^2) + n x_1^2 x_2^2.$$

En désignant alors par L_{ij}, $i \neq j$, l'expression

$$L_{ij} = V(V - x_i^2 - x_j^2) + n x_i^2 x_j^2,$$

il suit qu'on aura l'identité

$$-2\sum_{i<j}^{1\ldots n} L_{ij} x_i x_j = 2(n-1)\Sigma x_1^2 \Sigma x_1^2 x_2 x_3 -$$

$$- n(2\Sigma x_1^3 x_2^3 + 2\Sigma x_1^3 x_2^2 x_3 + 4\Sigma x_1^2 x_2^2 x_3 x_4 + 2\Sigma x_1^4 x_2 x_3).$$

L'identité (13) sera démontrée si nous démontrons l'identité suivante:

$$4(n-1)\Sigma x_1^2 \Sigma x_1^2 x_2^2 - n(\Sigma x_1^3 x_2^2 + 6\Sigma x_1^2 x_2^2 x_3^2) = \sum_{i<j}^{1\ldots n} L_{ij}(x_i^2 + x_j^2). \tag{14}$$

Puisque

$$\Sigma x_1^2 \Sigma x_1^2 x_2^2 = \Sigma x_1^4 x_2^2 + 3 \Sigma x_1^2 x_2^2 x_3^2,$$

l'égalité (14) prend la forme

$$(3n-4)\Sigma x_1^4 x_2^2 + 6(n-2)\Sigma x_1^2 x_2^2 x_3^2 = \sum_{i<j}^{1\ldots n} L_{ij}(x_i^2 + x_j^2). \tag{15}$$

Le terme $L_{12}(x_1^2 + x_2^2)$ dans le second membre de (15) est égal à

$$V(V-x_1^2-x_2^2)(x_1^2+x_2^2)+n x_1^2 x_2^2(x_1^2+x_2^2)-$$

$$-(x_1^2+x_2^2)^2(V-x_1^2-x_2^2)+(x_1^2+x_2^2)(V-x_1^2-x_2^2)^2+n x_1^2 x_2^2(x_1^2+x_2^2)=A+B,$$

où

$$A = 2x_1^2 x_2^2(x_3^2 + \cdots + x_n^2) + 2(x_1^2 + x_2^2) \sum_{i<j}^{3\ldots n} x_i^2 x_j^2,$$

$$B = (x_1^4 + x_2^4)(x_3^2 + \cdots + x_n^2) + (x_1^2 + x_2^2)(x_3^4 + \cdots + x_n^4) + n(x_1^4 x_2^2 + x_2^4 x_1^2).$$

Il est évident que tous les termes semblables à A dans le second membre de (15) auront pour somme la fonction symétrique $S = \Sigma x_1^2 x_2^2 x_3^2$, multipliée par un nombre entier et positif K. Mais dans chaque expression A il y a $2(n-2) + 4\binom{n-2}{2} = 2(n-2)^2$ termes de la fonction S et puisque le nombre des expressions A est égal à $\binom{n}{2}$, le nombre K sera égal à

$$K = \frac{2(n-2)^2\binom{n}{2}}{\binom{n}{3}} = 6(n-2).$$

Ainsi dans chaque expression de la forme B dans (15) il y a $2n-4 + 2n-4+2n = 6n-8$ termes de la fonction symétrique $S_1 = \Sigma x_1^4 x_2^2$. Mais le nombre des termes de cette fonction est égal à $n(n-1)$ et dans (15) on a $\binom{n}{2}$ termes de la forme B. Donc la somme des termes de type B dans le second membre de (15) sera égale à L, où le nombre L est déterminé par

$$L = \frac{(6n-8)\binom{n}{2}}{n(n-1)} = 3n-4.$$

Ainsi l'identité (15) est démontrée, de même que l'identité (13).

De (13) il découle que, quels que soient les nombres réels $x_1, x_2, \ldots, x_n$, on a toujours

$$U \geqq 0. \tag{16}$$

On a dans (16) le signe d'égalité seulement dans le cas où tous les nombres $x_1, x_2, \ldots, x_n$ sont égaux.

Par là c'est démontré qu'on aura toujours l'inégalité $U>0$ à l'exception du cas où $x_1 = x_2 = \cdots = x_n$. Dans le dernier cas le polynôme (11) prend la forme

$$c(x-x_1)^n \tag{17}$$

et le polynôme (5) devient

$$c(x-x_1)^{n-2}\left[(x-x_1)^2 - 2\varrho n \cos\varphi(x-x_1) + n(n-1)\varrho^2\right]. \tag{18}$$

Les zéros du polynôme (18) sont x_1 (de multiplicité $n-2$) et les nombres

$$n\varrho\left(\cos\varphi \pm \sqrt{\cos^2\varphi - \frac{n-1}{n}}\right).$$

Pour $\cos^2\varphi \geqq \dfrac{n-1}{n}$ tous les zéros de ce polynôme sont réels.

Supposons pour le moment que le polynôme (1) n'a pas la forme (17). De (1) nous obtenons un polynôme qui aura seulement des zéros réels et simples, en remplaçant chaque zéro z_k de multiplicité m par les m zéros simples $z_k, z_k(1+\varepsilon), z_k(1+2\varepsilon), \ldots, z_k(1+(m-1)\varepsilon)$, où $\varepsilon>0$ est suffisamment petit. Pour ε convenable on obtient ainsi un polynôme $f(x,\varepsilon)$, dont les zéros sont tous réels et simples. D'après les considérations ci-dessus il y aura un angle $\varphi_0'>0$, $\cos^2\varphi_0' < \dfrac{n-1}{n}$, tel que pour $0 \leqq \varphi \leqq \varphi_0'$ l'équation $F(x,\varepsilon)=0$ pour $f(x,\varepsilon)$ aura seulement des racines réelles. Puisque le polynôme $f(x,\varepsilon)$ tend vers $f(x)$, le polynôme $F(x,\varepsilon)$ tendra vers $F(x)$, lorsque $\varepsilon \to 0$ et les zéros du polynôme $F(x)$ seront tous réels pour $0 \leqq \varphi \leqq \varphi_0$, où φ_0 est déterminé par

$$\cos^2\varphi_0 = \frac{n-1}{n}.$$

On voit aussi que pour $\varphi < \varphi_0$ chaque zéro multiple du polynôme (5) doit être un zéro multiple du polynôme (1).

On voit tout de suite que la conclusion est vraie pour le polynôme (17) aussi et le théorème est complètement démontré.

Le théorème **1** se démontre par une voie inductive. Soient

$$z_k = \varrho_k(\cos\varphi_k + i\sin\varphi_k), \quad \bar{z}_k = \varrho_k(\cos\varphi_k - i\sin\varphi_k) \qquad (k=1,2,\ldots,p)$$

les zéros imaginaires du polynôme (1). Pour les arguments on suppose donc que

$$|\sin\varphi_k| < \frac{1}{\sqrt{n}}.$$

Le polynôme (2) a alors la forme

$$g(x)=g_1(x)(x-z_1)(x-\bar{z}_1)\cdots(x-z_p)(x-\bar{z}_p),$$

où $g_1(x)=b_0(x-\gamma_1)(x-\gamma_2)\cdots(x-\gamma_q)$ et les zéros $\gamma_1, \gamma_2, \ldots, \gamma_q$ du polynôme $g_1(x)$ sont les zéros réels de $g(x)$. D'après le théorème de HERMITE—POULAIN les zéros du polynôme $h(x)=g_1(D)f(x)$ sont tous réels et chaque zéro multiple de ce polynôme est un zéro multiple de $f(x)$. D'après la proposition **2** les zéros du polynôme

$$h_1(x)=z_1\bar{z}_1h(x)-(z_1+\bar{z}_1)h'(x)+h''(x)$$

seront tous réels et chaque zéro multiple de $h_1(x)$ sera un zéro multiple de $f(x)$. On aura la même proposition pour les zéros du polynôme

$$h_2(x)=z_2\bar{z}_2h_1(x)-(z_2+\bar{z}_2)h_1'(x)+h_1''(x)$$

etc. En suivant ce raisonnement on parviendra au polynôme (3) et le théorème **1** est ainsi démontré. On ne peut pas améliorer l'inégalité (4).

Considérons maintenant quelques conséquences du théorème **1**. Supposons que pour les arguments φ des zéros imaginaires du polynôme

$$f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n, \tag{19}$$

dont les coefficients sont réels, on a l'inégalité

$$|\sin\varphi|\leqq\frac{1}{\sqrt{n}}. \tag{20}$$

Il est évident que pour les zéros du polynôme

$$g(x)=x^nf\left(\frac{1}{x}\right)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n$$

on aura la même inégalité (20). D'après le théorème **1** le polynôme

$$g(D)x^n=n!\,a_0+(n-1)!\,a_1x+n(n-1)\cdots3a_2x^2+\cdots+a_nx^n$$

a seulement des zéros réels. Donc on a la proposition:

3. *Si les arguments des zéros du polynôme* (19), *dont les coefficients sont réels, satisfont à l'inégalité* (20), *la polynôme*

$$a_0+\frac{a_1}{1!}x+\frac{a_2}{2!}x^2+\cdots+\frac{a_n}{n!}x^n$$

a seulement des zéros réels.

De la même manière on démontre la proposition:

4. *Supposons que les arguments des zéros imaginaires du polynôme réel* (19) *satisfont à l'inégalité*

$$|\sin\varphi|\leqq\frac{1}{\sqrt{p}},$$

où p est un nombre entier et positif. Alors, si $p \geqq n$, le polynôme

$$\frac{a_0}{(p-n)!}+\frac{a_1}{(p+1-n)!}x+\frac{a_2}{(p+2-n)!}x^2+\cdots+\frac{a_n}{p!}x^n$$

a tous ses zéros réels. Si $p<n$, le polynôme

$$a_{n-p}+\frac{a_{n-p+1}}{1!}x+\frac{a_{n-p+2}}{2!}x^2+\cdots+\frac{a_n}{p!}x^p$$

a tous ses zéros réels.

On doit à SCHUR [1] le théorème suivant:

Si les zéros du polynôme

(21) $$f(x)=a_0+a_1x+\cdots+a_mx^m \qquad (a_m\neq 0)$$

sont réels et les zéros du polynôme

(22) $$\varphi(x)=b_0+b_1x+\cdots+b_nx^n \qquad (b_n\neq 0)$$

sont réels et du même signe, les zéros du polynôme

(23) $$a_0b_0+1!\,a_1b_1x+2!\,a_2b_2x^2+\cdots+k!\,a_kb_kx^k \qquad (k=\min(m,n))$$

sont aussi réels.

MALO [2] a démontré plus tôt le théorème:

Si les zéros du polynôme (21) $(a_m\neq 0)$ sont réels et les zéros du polynôme (22) $(b_n\neq 0)$ sont réels et du même signe, les zéros du polynôme $a_0b_0+a_1b_1x+\cdots+a_kb_kx^k$ $(k=\min(m,n))$ sont tous réels.

En se basant sur le théorème **1**, nous démontrons la généralisation suivante du théorème de SCHUR:

5. *Supposons que les zéros du polynôme* (21) $(a_m\neq 0)$ *sont réels et que les arguments des zéros du polynôme réel* (22) $(b_n\neq 0)$ *satisfont tous à l'inégalité $-\alpha\leqq\varphi\leqq\alpha$, ou à l'inégalite $\pi-\alpha\leqq\varphi\leqq\pi+\alpha$, où $\alpha>0$ est l'angle déterminé par* $\sin\alpha=\frac{1}{\sqrt{m}}$. *Alors le polynôme* (23) *à seulement des zéros réels.*

Considérons d'abord le cas où $a_0b_0\neq 0$. On peut se borner au cas de l'inégalité $\pi-\alpha\leqq\varphi\leqq\pi+\alpha$. Alors les coefficients du polynôme (22) seront du même signe qu'on peut supposer positif. Nous suivrons la marche de la démonstration de SCHUR. Soit z un nombre réel et considérons d'abord le cas où $m\leqq n$. Le polynôme

(24) $$F(x)=b_0f(x)+b_1zf'(x)+\cdots+b_mz^mf^{(m)}(x)$$

a la forme

$$F(x)=P_0(z)+P_1(z)\frac{x}{1!}+P_2(z)\frac{x^2}{2!}+\cdots+P_m(z)\frac{x^m}{m!},$$

où

$$P_0(z) = a_0 b_0 + 1!\, a_1 b_1 z + 2!\, a_2 b_2 z^2 + \cdots + m!\, a_m b_m z^m$$

et

$$P_\mu(z) = \mu!\, a_\mu b_0 + (\mu+1)!\, a_{\mu+1} b_1 z + \cdots + m!\, a_m b_{m-\mu} z^{m-\mu} \qquad (1 \leqq \mu \leqq m).$$

Si pour les arguments des zéros du polynôme (22) on a $|\sin\varphi| < \frac{1}{\sqrt{m}}$, la même proposition sera valable pour les zéros du polynôme

$$b_0 + b_1 z x + b_2 z^2 x^2 + \cdots + b_n z^n x^n.$$

Donc d'après le théorème **1** le polynôme (24) aura seulement des zéros réels, quel que soit le nombre réel z. Les polynômes $P_0(z)$ et $P_1(z)$ ne peuvent pas avoir des zéros communs puisque dans le cas contraire $x=0$ sera un zéro multiple de (24) et donc un zéro pareil du polynôme (21), ce qui est impossible à cause de $a_0 \neq 0$.

Des conséquences connues du théorème de Descartes il suit que les polynômes

(25) $$P_0(z), P_1(z), P_2(z), \ldots, P_m(z)$$

forment une suite de Sturm. Pour $z = -\infty$ dans suite (25) il n'y a pas des variations et pour $z = \infty$ cette suite a m variations. Donc d'après le théorème généralisé de Sturm le polynôme $P_0(z)$ doit avoir seulement des zéros réels et le rapport $P_1(z)/P_0(z)$ doit passer de positif en négatif m fois en s'annulant lorsque z croît de $-\infty$ à ∞. Donc il suit encore que les zéros du polynôme $P_0(z)$ sont tous simples. Le cas $|\sin\varphi| \leqq \frac{1}{\sqrt{m}}$ est un cas limite de $|\sin\varphi| < \frac{1}{\sqrt{m}}$ et la réalité des zéros du polynôme est garantie par le théorème classique de Hurwitz.

Considérons maintenant le cas $m > n$. Au lieu de polynôme $\varphi(x)$ nous prendrons le polynôme

$$(1+\varepsilon x)^{m-n} \varphi(x) = b_0(\varepsilon) + b_1(\varepsilon) x + \cdots + b_m(\varepsilon) x^m,$$

où ε est un nombre positif. Le polynôme

$$a_0 b_0(\varepsilon) + 1!\, a_1 b_1(\varepsilon) x + \cdots + n!\, a_n b_n(\varepsilon) x^n + \cdots + m!\, a_m b_m(\varepsilon) x^m$$

aura tous ses zéros réels. En faisant ici ε tendre vers zéro, on obtient le polynôme

$$a_0 b_0 + 1!\, a_1 b_1 x + \cdots + n!\, a_n b_n x^n,$$

qui doit avoir, grâce au théorème de Hurwitz, seulement des zéros réels. Si $a_0 = b_0 = 0$, nous pouvons écrire

$$f(x) = x^p f_1(x), \quad \varphi(x) = x^q \varphi_1(x),$$

où $f_1(x)$ et $\varphi_1(x)$ sont des polynômes différents de zéro pour $x=0$. Comme ci-dessus, nous introduisons les polynômes

$$f_\delta(x)=(x+\delta)^p f_1(x), \quad \varphi_\delta(x)=(x+\delta)^q \varphi_1(x)$$

et le polynôme composé de ces deux polynômes tendra vers le polynôme composé (23).

6. (Généralisation du théorème de Malo.) *Supposons que les arguments des zéros imaginaires du polynôme réel* (21) *satisfont à l'inégalité* $|\sin\varphi| \leqq \dfrac{1}{\sqrt{m}}$ *et que les arguments des zéros imaginaires du polynôme réel* (22), *dont tous les coefficients sont du même signe, satisfont à la même inégalité. Alors le polynôme*

$$a_0 b_0 + a_1 b_1 x + a_2 b_2 x^2 + \cdots + a_k b_k x^k \qquad (k=\min(m, n))$$

a seulement des zéros réels.

Pour la démonstration on applique les théorèmes **3** et **5**.

(Reçu le 1 mars 1960.)

Ouvrages cités

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[2] E. Malo, Note sur les équations algébriques dont toutes les racines sont réelles, *Journal de Math. spéciales* (4), **4** (1895), p. 7.

ON SUMS OF POWERS OF COMPLEX NUMBERS

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(*Presented by* P. TURÁN)

1. Let $z_1, \ldots, z_n$ be complex numbers such that

$$1 = z_1 \geqq |z_2| \geqq \cdots \geqq |z_n|, \tag{1}$$

and write

$$s_k = \sum_{m=1}^{n} z_m^k, \quad s = \max_{1 \leqq k \leqq n} |s_k|. \tag{2}$$

The problem has been posed by TURÁN[1] of finding a positive lower bound for s, valid for all choices of $z_1, \ldots, z_n$ subject to (1). The lower bound $(\log 2) \Big/ \sum_1^n \frac{1}{m}$, due to TURÁN, was improved by N. G. DE BRUIJN to $C \log\log n / \log n$, for some $C > 0$, and sufficiently large n. It was subsequently shown by S. UCHIYAMA[2] that C could be taken arbitrarily close to 1. The aim of the present note is to verify the conjecture that s has a positive lower bound independent of n. Without any suggestion that this is a precise value, we show that

$$s > 1/6. \tag{3}$$

2. As in [2], we use the result that

$$\exp\left\{-\sum_1^n m^{-1} s_m y^m\right\} = \prod_{r=1}^{n} (1 - z_r y) + \sum_{m=n+1}^{\infty} c_m y^m, \tag{4}$$

for all y; here the c_m are functions of $z_1, \ldots, z_n$. In particular, writing

$$g(\theta) = -\sum_1^n m^{-1} s_m e^{mi\theta}$$

we have

$$e^{g(\theta)} = \prod_{r=1}^{n} (1 - z_r e^{i\theta}) + \sum_{n+1}^{\infty} c_m e^{mi\theta}, \tag{5}$$

[1] P. TURÁN, *Eine neue Methode in der Analysis und deren Anwendungen* (Budapest, 1953, and Peking, 1956). — P. TURÁN, Über die Potenzsummen komplexer Zahlen, *Archiv der Math.*, **9** (1958), pp. 59—64.

[2] S. UCHIYAMA, Sur les sommes de puissances des nombres complexes, *Acta Math. Acad. Sci. Hung.*, **9** (1958), pp. 275—278.

and the special case $\theta=0$ gives, since $z_1=1$,

$$e^{g(0)}=\sum_{n+1}^{\infty} c_m. \tag{6}$$

We now evaluate the c_m as Fourier coefficients of $e^{g(\theta)}$, according to (5). We get

$$c_m=(2\pi)^{-1}\int_{-\pi}^{\pi} e^{-mi\theta}e^{g(\theta)}d\theta.$$

Integration by parts gives

$$c_m=[-(2\pi mi)^{-1}e^{-mi\theta+g(\theta)}]_{-\pi}^{\pi}+(2\pi mi)^{-1}\int_{-\pi}^{\pi} g'(\theta)e^{-mi\theta+g(\theta)}d\theta,$$

and here the integrated term plainly vanishes. Substituting in (6) we get

$$e^{g(0)}=(2\pi i)^{-1}\sum_{n+1}^{\infty} m^{-1}\int_{-\pi}^{\pi} g'(\theta)e^{-mi\theta+g(\theta)}d\theta=(2\pi i)^{-1}\int_{-\pi}^{\pi} g'(\theta)e^{g(\theta)}h(\theta)d\theta \tag{7}$$

where

$$h(\theta)=\sum_{n+1}^{\infty} m^{-1}e^{-mi\theta}$$

or

$$1=(2\pi i)^{-1}\int_{-\pi}^{\pi} g'(\theta)e^{g(\theta)-g(0)}h(\theta)d\theta. \tag{8}$$

In deriving (7) from the preceding equation we have inverted the order of summation and integration. This may be justified by the theory of mean-square convergence. Alternatively, it would be possible to avoid this difficulty by replacing $h(\theta)$ by a sufficiently long partial sum of the series which defines it; this, however, would complicate the working.

3. Taking absolute values in (8) and using Schwarz's inequality, we have

$$1\leqq(2\pi)^{-2}\int_{-\pi}^{\pi}|g'(\theta)|^2d\theta\int_{-\pi}^{\pi}|e^{g(\theta)-g(0)}|^2|h(\theta)|^2d\theta. \tag{9}$$

The first integral on the right is readily estimated. Since

$$g'(\theta)=-i\sum_{1}^{n} s_m e^{mi\theta},$$

the Parseval equality shows that

$$\int_{-\pi}^{\pi}|g'(\theta)|^2d\theta=2\pi\sum_{1}^{n}|s_m|^2\leqq 2\pi n s^2. \tag{10}$$

For the second integral in (9), we consider first the interval $-\pi/n \leqq \theta \leqq \pi/n$. Here

$$|g(\theta)-g(0)| \leqq \sum_1^n m^{-1}|s_m|\,|e^{mi\theta}-1| \leqq \sum_1^n m^{-1}|s_m|\,|m\theta| \leqq |\theta n s| \leqq \pi s$$

in this interval. Hence

$$\text{(11)} \qquad \begin{aligned} \int_{-\pi/n}^{\pi/n} |e^{g(\theta)-g(0)}|^2 |h(\theta)|^2 d\theta &\leqq e^{2\pi s} \int_{-\pi/n}^{\pi/n} |h(\theta)|^2 d\theta < e^{2\pi s} \int_{-\pi}^{\pi} |h(\theta)|^2 d\theta = \\ &= 2\pi e^{2\pi s} \sum_{n+1}^{\infty} m^{-2} < 2\pi e^{2\pi s} n^{-1}. \end{aligned}$$

For $\pi/n \leqq \theta \leqq \pi$ we estimate $g(\theta)-g(0)$ as follows:

$$|g(\theta)-g(0)| \leqq \sum_{m \leqq \pi/\theta} m^{-1}|s_m|\,|m\theta| + \sum_{\pi/\theta < m \leqq n} m^{-1}|s_m|\,2 \leqq \pi s + 2s(\log(n\theta/\pi)+1).$$

Hence

$$|e^{g(\theta)-g(0)}| \leqq e^{s(\pi+2)}(n\theta/\pi)^{2s}.$$

We also have to estimate $h(\theta)$. We have

$$(1-e^{-i\theta})h(\theta) = (n+1)^{-1}e^{-(n+1)i\theta} - \sum_{r=2}^{\infty} e^{-(n+r)i\theta}\{(n+r-1)^{-1}-(n+r)^{-1}\}$$

whence

$$|(1-e^{-i\theta})h(\theta)| \leqq 2/(n+1).$$

Hence

$$|h(\theta)| \leqq (n+1)^{-1}\left|\operatorname{cosec}\frac{1}{2}\theta\right| < \pi/(n\theta)$$

if $\pi/n \leqq \theta \leqq \pi$.

Hence

$$\text{(12)} \qquad \begin{aligned} \int_{\pi/n}^{\pi} |e^{g(\theta)-g(0)}|^2 |h(\theta)|^2 d\theta &\leqq e^{2s(\pi+2)} \int_{\pi/n}^{\pi} (n\theta/\pi)^{4s-2} d\theta = (\pi/n)e^{2s(\pi+2)} \int_1^n \varphi^{4s-2} d\varphi < \\ &< \pi e^{2s(\pi+2)} n^{-1}(1-4s)^{-1}, \end{aligned}$$

assuming that $s<1/4$; this is enough since we aim to prove only that $s>1/6$. Since a similar bound holds for the integral over $(-\pi, -\pi/n)$, it follows from (11) and (12) that

$$\int_{-\pi}^{\pi} |e^{g(\theta)-g(0)}|^2 |h(\theta)|^2 d\theta < 2\pi n^{-1} e^{2\pi s}\{1+e^{4s}(1-4s)^{-1}\}.$$

Inserting this and (10) in (9) we get

$$\text{(13)} \qquad 1 < s^2 e^{2\pi s}\{1+e^{4s}(1-4s)^{-1}\}.$$

Since the expression on the right tends to zero with s, and is independent of n, we deduce that s has a positive lower bound which is independent of n. That (13) implies (3) follows from the fact that the inequality (13) is false for $s = 1/6$, together with the fact that the function on the right is monotonic increasing in $0 < s < 1/4$.

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ON GENERALIZATIONS OF AN INEQUALITY DUE TO PÓLYA AND SZEGŐ

By
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(*Presented by* P. TURÁN)

1. In a recent paper W. GREUB and W. RHEINBOLDT [1] prove the inequality

$$(1)\qquad \sum_{k=1}^{\infty} a_k^2 \xi_k^2 \sum_{k=1}^{\infty} b_k^2 \xi_k^2 \leqq \frac{(M_1 M_2 + m_1 m_2)^2}{4 M_1 M_2 m_1 m_2} \left[\sum_{k=1}^{\infty} a_k b_k \xi_k^2\right]^2$$

where $\xi_1^2 + \xi_2^2 + \cdots + \xi_k^2 + \cdots < \infty$ and

$$(2)\qquad 0 < m_1 \leqq a_k \leqq M_1, \quad 0 < m_2 \leqq b_k \leqq M_2.$$

They show that it is equivalent to the inequality

$$(2')\qquad \sum_{k=1}^{\infty} \gamma_k \xi_k^2 \sum_{k=1}^{\infty} \frac{1}{\gamma_k} \xi_k^2 \leqq \frac{(M+m)^2}{4Mm} \left[\sum_{k=1}^{\infty} \xi_k^2\right]^2 \qquad (\Sigma \gamma_k < \infty,\ 0 < m \leqq \gamma_k \leqq M)$$

which the authors attribute to L. V. KANTOROVICH[1] and is a generalization of an inequality due to PÓLYA and SZEGŐ [2]. Their proof is done *via* the theory of linear operators. However, owing to the elementary character of the inequality (1) it is of some interest to have a proof involving only the elements of analysis. (In part 3 of this paper it will be seen that the same argument suffices to prove that general inequality in Hilbert space from which GREUB and RHEINBOLDT derived (1).)

The proof given here will enable us to discuss the case of equality in (1). By putting

$$\alpha_k = (m_1 M_1)^{-\frac{1}{2}} a_k, \quad \beta_k = (m_2 M_2)^{-\frac{1}{2}} b_k, \quad a = (M_1/m_1)^{\frac{1}{2}}, \quad b = (M_2/m_2)^{\frac{1}{2}}$$

it is seen that the above statement is equivalent to the following assertion:

[1] The inequality (2') is contained in an integral inequality obtained by P. SCHWEITZER [3] who has shown that if $0 < m \leqq F(x) \leqq M$ in $p \leqq x \leqq q$, then

$$\int_p^q F(x)\,dx \int_p^q \frac{1}{F(x)}\,dx \leqq \frac{(M+m)^2}{4Mm}.$$

By taking $p = 0$, $q = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 + \cdots$, $F(x) = \gamma_1$ if $0 \leqq x < \xi_1^2$, $F(x) = \gamma_k$ if $\xi_1^2 + \xi_2^2 + \cdots + \xi_{k-1}^2 \leqq x < \xi_1^2 + \xi_2^2 + \cdots + \xi_k^2$ $(k = 2, 3, \ldots)$, we have (2'), provided that $F(x)$ and $[F(x)]^{-1}$ are Riemann integrable which is true if f. i. the γ_k's form an increasing sequence of numbers.

If $a^{-1} \leqq \alpha_k \leqq a$, $b^{-1} \leqq \beta_k \leqq b$ *and* $\xi_1^2 + \xi_2^2 + \cdots + \xi_k^2 \cdots < \infty$, *then*

$$[(ab)+(ab)^{-1}]^2 \left(\sum_{k=1}^{\infty} \alpha_k \beta_k \xi_k^2 \right)^2 - 4 \sum_{k=1}^{\infty} \alpha_k^2 \xi_k^2 \sum_{k=1}^{\infty} \beta_k^2 \xi_k^2 \geqq 0. \tag{3}$$

Put now

$$l_{1k}(x) = \alpha_k x - ab\beta_k \quad \text{and} \quad l_{2k}(x) = ab\alpha_k x - \beta_k.$$

Then

$$l_{1k}(1) = \alpha_k - ab\beta_k \leqq a - abb^{-1} = 0 \tag{4_1}$$

and

$$l_{2k}(1) = ab\alpha_k - \beta_k \geqq aba^{-1} - b = 0 \tag{4_2}$$

with signs of equality only if

$$(5_1) \qquad \alpha_k = a,\ \beta_k = b^{-1} \quad \text{and} \quad (5_2) \qquad \alpha_k = a^{-1},\ \beta_k = b,$$

respectively.

Further if

$$f(x) = \sum_{k=1}^{\infty} \frac{\xi_k^2}{ab} l_{1k}(x) l_{2k}(x) = \sum_{k=1}^{\infty} \alpha_k^2 \xi_k^2 x^2 - [ab + (ab)^{-1}] \sum_{k=1}^{\infty} \alpha_k \beta_k \xi_k^2 x + \sum_{k=1}^{\infty} \beta_k^2 \xi_k^2,$$

then $f(x) \to +\infty$ as $x \to \infty$ and $f(1) \leqq 0$. Hence the quadratic equation $f(x) = 0$ has at least one real root, and therefore its discriminant, the left-hand side of (3) is non-negative.

2. For finding the conditions of equality in (3) we define two sets of natural numbers K_1 and K_2 as follows: if $k \in K_i$, then (5_i) holds $(i = 1, 2)$. The left-hand side of (3) vanishes if and only if $f(1)$ and $f'(1)$ vanish simultaneously. $f(1) = 0$ if and only if $K_1 \cup K_2$ is the set K of natural numbers. In this case an easy computation shows that

$$f'(1) = \frac{a^2 b^2 - 1}{a^2 b^2} \left(a^2 \sum_{k \in K_1} \xi_k^2 - b^2 \sum_{k \in K_2} \xi_k^2 \right)$$

and $f'(1)$ (consequently the left of (3)) vanishes if either $a^2 b^2 = 1$ (or what amounts to the same $a = b = 1$) or

$$a^2 \sum_{k \in K_1} \xi_k^2 = b^2 \sum_{k \in K_2} \xi_k^2 \qquad (K_1 \cup K_2 = K).$$

3. GREUB and RHEINBOLDT [1] found a generalization of inequality (1) in Hilbert space which they call the *generalized Pólya—Szegő inequality:*

Given two permutable, linear and self-adjoint operators A and B of the Hilbert space H which fulfil the conditions[2]

$$0 < m_1 I \leqq A \leqq M_1 I, \quad 0 < m_2 I \leqq B \leqq M_2 I,$$

[2] I denotes the identity operator in H. $C \leqq D$ means that $(Cx, x) \leqq (Dx, x)$ for all $x \in H$. C is positive if $(Cx, x) \geqq 0$ for all $x \in H$.

then

$$(6) \qquad (Ax, Ax)(Bx, Bx) \leqq \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2} (Ax, Bx)^2$$

for all $x \in H$.

The device used in showing (1) can be applied to yield a short proof of this inequality, the main result of the paper of GREUB and RHEINBOLDT. We will use the fact that the product of two permutable positive bounded linear self-adjoint operators is positive, too.

There is no loss of generality in taking $M_1 = a$, $m_1 = a^{-1}$, $M_2 = b$, $m_2 = b^{-1}$, so that it is enough to show that in this case

$$(7) \qquad [ab + (ab)^{-1}]^2 (Ax, Bx)^2 - 4(Ax, Ax)(Bx, Bx) \geqq 0.$$

PROOF. Consider the quadratic form

$$f(\lambda) = -(Ax, Ax)\lambda^2 + [ab + (ab)^{-1}](Ax, Bx)\lambda - (Bx, Bx)$$

for fixed x. We have $f(1) = (Mx, Nx)$ with $M = abB - A$ and $N = A - (ab)^{-1}B$. Both M and N are positive and bounded, for

$$(Mx, x) = ab(Bx, x) - (Ax, x) \geqq (abb^{-1} - a)(x, x) = 0$$

and

$$(Nx, x) = (Ax, x) - (ab)^{-1}(Bx, x) \geqq (a^{-1} - (ab)^{-1}b)(x, x) = 0,$$

and so the operator NM is positive. Hence $f(1) = (NMx, x) \geqq 0$.

On the other hand, $f(\lambda) \to -\infty$ if $\lambda \to \infty$ as (Ax, Ax) is positive. Hence the quadratic equation $f(\lambda) = 0$ has at least one real root and (7) holds.

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SOME INTERPOLATORY PROPERTIES OF HERMITE POLYNOMIALS

By

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(*Presented by* P. TURÁN)

1. Professor TURÁN and his colleagues in a series of papers on interpolation have discussed i) the problem of existence and uniqueness, ii) the problem of explicit representation and iii) the problem of convergence for (0, 2)-interpolation by taking as abscissae the zeros of $\Pi_n(x)=(1-x^2)P'_{n-1}(x)$ where $P_{n-1}(x)$ is the Legendre polynomial of degree $<n-1$. By (0, 2)-interpolation we mean the construction of a polynomial of degree $\leqq 2n-1$, when the value of the function and its second derivative at the zeros of $\Pi_n(x)$ are prescribed.

Later on SAXENA and SHARMA [3] have studied the aforesaid problems for (0, 1, 3)-interpolation taking the same abscissae as those used by P. TURÁN. Later SAXENA [4] has extended the results to (0, 1, 2, 4)-interpolation.

The object of this note is to consider the problem of existence and uniqueness and of explicit representation for (0, 2)- and (0, 1, 3)-interpolation, respectively, choosing the abscissae as the zeros of $H_n(x)$, the Hermite polynomial of degree n, which are given by

$$\infty > x_{1,n} > x_{2,n} > \cdots > x_{n,n} > -\infty.$$

2. We shall prove the following theorems:

THEOREM I. (Case (0, 2).) *If $n=2k$, then to prescribed values y_ν and y_ν^* there is a uniquely determined polynomial $g(x)$ of degree $\leqq 2n-1$ such that*

$$g(x_{\nu,n})=y_\nu \quad \text{and} \quad g''(x_{\nu,n})=y_\nu^* \qquad (\nu=1,2,\ldots,n) \tag{2.1}$$

if $x_{\nu,n}$'s stand for the zeros of $H_n(x)$.

This means, of course, that in case

$$y_\nu=y_\nu^*=0 \qquad (\nu=1,2,\ldots,n;\, n \text{ even}) \tag{2.2}$$

the only solution of (2. 1) is $g(x)\equiv 0$.

THEOREM II. (Case (0, 1, 3).) *If $n=2k$, then to prescribed values $y_\nu, y_\nu^*, y_\nu^{**}$ there is a uniquely determined polynomial $f(x)$ of degree $\leqq 3n-1$ such that*

$$f(x_{\nu,n})=y_\nu, \quad f'(x_{\nu,n})=y_\nu^* \quad \text{and} \quad f'''(x_{\nu,n})=y_\nu^{**} \qquad (\nu=1,2,\ldots,n) \tag{2.3}$$

(if $x_{\nu,n}$'s stand for the zeros of $H_n(x)$.

This means, of course, that in case

$$y_\nu = y^*_\nu = y^{**}_\nu = 0 \qquad (\nu = 1, 2, \ldots, n;\ n \text{ even}) \tag{2.4}$$

the only solution of (2. 3) is $f(x) \equiv 0$.

3. Preliminaries. In this section we shall give certain well-known formulae which we shall use later on.

$$H''_n(x) - 2xH'_n(x) + 2nH_n(x) = 0 \tag{3.1}$$

is the well-known differential equation satisfied by $H_n(x)$. At $x = x_j$,

$$H''_n(x_j) = 2x_j H'_n(x_j) \qquad (j = 1, 2, \ldots, n). \tag{3.2}$$

We shall denote by $l_\nu(x)$ the fundamental polynomials of the Lagrange interpolation based on the x_ν's, i. e.

$$l_\nu(x) = \frac{H_n(x)}{(x - x_\nu)H'_n(x_\nu)}. \tag{3.3}$$

From this it is easy to see that

$$l_\nu(x_\nu) = 1, \quad l_\nu(x_j) = 0, \tag{3.4}$$

$$l'_\nu(x_\nu) = x_\nu, \quad l'_\nu(x_j) = \frac{H'_n(x_j)}{H'_n(x_\nu)(x_j - x_\nu)}, \tag{3.5}$$

$$\begin{cases} l''_\nu(x_\nu) = \dfrac{4x_\nu^2 + 2(1-n)}{3}, \\[2ex] l''_\nu(x_j) = \dfrac{2H'_n(x_j)}{(x_j - x_\nu)H'_n(x_\nu)} \left\{ x_j - \dfrac{1}{x_j - x_\nu} \right\}. \end{cases} \tag{3.6}$$

Besides this we shall also make use of the fact that

$$\sum_{r=0}^{n} \frac{H_r(x)H_r(y)}{2^r r!} = \frac{1}{2^{n+1} n!} \, \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{x - y}. \tag{3.7}$$

Taking $y = x_\nu$ and replacing $n+1$ by n in (3. 7), we get

$$\sum_{r=0}^{n-1} \frac{H_r(x)H_r(x_\nu)}{2^r r!} = \frac{1}{2^n (n-1)!} \, \frac{H_n(x)H_{n-1}(x_\nu)}{(x - x_\nu)}. \tag{3.8}$$

We also require the following well-known properties:

$$H'_n(x) = 2nH_{n-1}(x), \tag{3.9}$$

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x) \qquad (n = 2, 3, 4, \ldots), \tag{3.10}$$

$$H_{2m}(0) = \frac{(-1)^m (2m)!}{m!}, \qquad H'_{2m+1}(0) = (-1)^m \frac{(2m+2)!}{(m+1)!}. \tag{3.11}$$

Also $H_0(x) = 1$, and $H_1(x) = 2x$.

4. Proof of Theorem I. Since $g(x)$ is a polynomial of degree $\leqq 2n-1$, we have

$$g(x)=H_n(x)r_{n-1}(x) \tag{4. 1}$$

where $r_{n-1}(x)$ is a polynomial of degree $\leqq n-1$, so that the first part of condition (2. 1) is obviously satisfied and from the second we have

$$g''(x_j)=H_n''(x_j)r_{n-1}(x_j)+2H_n'(x_j)r_{n-1}'(x_j)=0.$$

Since x_j's are the simple zeros of $H_n(x)$, we get

$$r_{n-1}'(x_j)+x_jr_{n-1}(x_j)=0,$$

with the help of (3. 2). Hence

$$r_{n-1}'(x)+xr_{n-1}(x)=cH_n(x) \tag{4. 2}$$

where c is a constant.

Let the solution of this differential equation be

$$r_{n-1}(x)=\sum_{\nu=0}^{n-1}c_\nu H_\nu(x).$$

Substituting this in (4. 2), we get with the help of (3. 9) and (3. 10),

$$3\sum_{\nu=0}^{n-2}(\nu+1)c_{\nu+1}H_\nu(x)+\frac{1}{2}\sum_{\nu=1}^{n}c_{\nu-1}H_\nu(x)=cH_n(x). \tag{4. 3}$$

Now equating the coefficients of $H_\nu(x)$ on both sides, we get

$$3c_1=0,$$

$$3(\nu+1)c_{\nu+1}+\frac{1}{2}c_{\nu-1}=0 \qquad (\nu=1,2,\ldots,n-2),$$

$$\frac{1}{2}c_{n-2}=0 \quad \text{and} \quad \frac{1}{2}c_{n-1}=c.$$

Since $c_1=0$ and n is even, we have

$$c_1=c_3=c_5=\cdots=c_{n-1}=0.$$

Also

$$c_{n-2}=c_{n-4}=\cdots=c_2=c_0=0.$$

Hence all the c_ν's are zero. So the solution is $g(x)\equiv 0$.

When n is odd, we get from (4. 3)

$$3c_1=0,$$

$$3(\nu+1)c_{\nu+1}+\frac{1}{2}c_{\nu-1}=0 \quad \text{for} \quad \nu=1,2,\ldots,n-2.$$

Therefore

$$c_1 = c_3 = c_5 = \cdots = c_{n-2} = 0$$

and $c_0, c_2, c_4, \ldots, c_{n-1}$ can be determined and are non-zero. Therefore, when n is odd, there are an infinity of solutions if they exist.

Theorem II can be proved on the same lines.

5. Problem of explicit representation. (Case (0, 2).) Given distinct points

$$\infty > x_1 > x_2 \cdots > x_n > -\infty$$

and arbitrary numbers

$$\alpha_1, \alpha_2, \ldots, \alpha_n; \ \beta_1, \beta_2, \ldots, \beta_n;$$

it is to be decided whether or not there is a polynomial $R_n(x)$ of degree $\leqq 2n-1$ such that

$$R_n(x_\nu) = \alpha_\nu, \qquad R_n''(x_\nu) = \beta_\nu \qquad (\nu = 1, 2, \ldots, n).$$

Throughout this paper we shall take n to be even. So for $n = 2k$ we have

$$R_{2k}(x) = \sum_{\nu=1}^{2k} \alpha_\nu r_\nu(x) + \sum_{\nu=1}^{2k} \beta_\nu \varrho_\nu(x) \tag{5. 1}$$

where $r_\nu(x)$ and $\varrho_\nu(x)$ are the fundamental polynomials of the first and second kind of (0, 2)-interpolation, of degree $\leqq 2n-1 = 4k-1$, uniquely determined by the following conditions:

$$r_\nu(x_j) = \begin{cases} 1 & \text{for } j = \nu \\ 0 & \text{for } j \neq \nu \end{cases} \qquad (j = 1, 2, \ldots, n), \tag{5. 2}$$

$$r_\nu''(x_j) = 0 \qquad (j = 1, 2, \ldots, n) \tag{5. 3}$$

and

$$\varrho_\nu(x_j) = 0 \qquad (j = 1, 2, \ldots, n), \tag{5. 4}$$

$$\varrho_\nu''(x_j) = \begin{cases} 1 & \text{for } j = \nu \\ 0 & \text{for } j \neq \nu \end{cases} \qquad (j = 1, 2, \ldots, n). \tag{5. 5}$$

We then have the following

THEOREM III. *For the fundamental polynomials $r_\nu(x)$ and $\varrho_\nu(x)$ the following explicit forms hold true:*

$$\begin{aligned} r_\nu(x) = l_\nu^2(x) + \frac{e^{-\frac{x^2}{2}} H_n(x)}{H_n'(x_\nu)} \Bigg[A \int_0^x e^{\frac{t^2}{2}} H_n(t)\,dt \, + \\ + B \int_0^x \frac{e^{\frac{t^2}{2}} H_n(t)}{t - x_\nu}\,dt - \frac{1}{H_n''(x_\nu)} \int_0^x \frac{H_n'(t) - (Ct + D) l_\nu(t)}{(t - x_\nu)^2} e^{\frac{t^2}{2}}\,dt + K \Bigg] \end{aligned} \tag{5. 6}$$

$$(\nu = 1, 2, \ldots, n),$$

where A, B, C, D *and* K *are constants given by*

$$(5.7)\qquad A=-\frac{\left(-\frac{4}{3}\right)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{2H'_n(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{\left(-\frac{3}{4}\right)^r(n-2r-1-2x_\nu^2)}{r!}H_{2r}(x_\nu),$$

$$(5.8)\qquad B=-\frac{2x_\nu^2}{H'_n(x_\nu)},$$

$$(5.9)\qquad C=x_\nu H'_n(x_\nu),$$

$$(5.10)\qquad D=(1-x_\nu^2)H'_n(x_\nu),$$

(5. 11)

$$K=\frac{2^n(n-1)!}{H'_n(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{(-1)^r(n-2r-2-2x_\nu^2)}{2^{2r+1}\Gamma\left(r+\frac{3}{2}\right)}H_{2r+1}(x_\nu)\sum_{i=1}^{r+1}3^{i-1}\frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)},$$

$$(5.12)\qquad \varrho_\nu(x)=\frac{e^{-\frac{x^2}{2}}H_n(x)}{2H'_n(x_\nu)}\left[a\int_0^x e^{\frac{t^2}{2}}H_n(t)\,dt+\frac{1}{H'_n(x_\nu)}\int_0^x\frac{e^{\frac{t^2}{2}}H_n(t)}{t-x_\nu}\,dt+b\right],$$

where a *and* b *are constants given by*

$$(5.13)\qquad a=-\frac{\left(-\frac{4}{3}\right)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{2H'_n(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\left(-\frac{3}{4}\right)^r\frac{H_{2r}(x_\nu)}{r!}$$

and

$$(5.14)\qquad b=\frac{2^n(n-1)!}{H'_n(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{(-1)^rH_{2r+1}(x_\nu)}{2^{2r+1}\Gamma\left(r+\frac{3}{2}\right)}\sum_{i=1}^{r+1}3^{i-1}\frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)}.$$

Other equivalent forms for (5. 13), (5. 14), (5. 7) and (5. 11) are the following:

$$(5.13\text{a})\qquad a=-\frac{\left(-\frac{1}{12}\right)^n}{\sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right)H'_n(x_\nu)}\int_{-\infty}^{\infty}\frac{H_n(it)}{it-x_\nu}e^{-\frac{t^2}{2}}\,dt,$$

$$(5.14\text{a})\qquad b=-\frac{i}{\sqrt{2\pi}\,H'_n(x_\nu)}\int_0^1\frac{du}{\sqrt{1-u^2}}\int_{-\infty}^{\infty}te^{-\frac{t^2}{2}}\frac{H_n(iut)}{iut-x_\nu}\,dt,$$

$$\text{(5.7a)}\quad A=\frac{\left(-\frac{1}{12}\right)^{\frac{n}{2}}}{\sqrt{2}\,\Gamma\left(\frac{n+1}{2}\right)H_n'(x_\nu)}\left[2x_\nu^2\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\frac{H_n(it)}{it-x_\nu}\,dt+\right.$$
$$\left.+\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\frac{H_n'(it)-(itx_\nu+1-x_\nu^2)l_\nu(it)H_n'(ix_\nu)}{(it-x_\nu)^2}\,dt\right],$$

$$\text{(5.11a)}\quad K=\frac{i}{\sqrt{2\pi}\,H_n'(x_\nu)}\int_0^1\frac{du}{\sqrt{1-u^2}}\left[\int_{-\infty}^{\infty}te^{-\frac{t^2}{2}}\left\{2x_\nu^2\frac{H_n(iut)}{iut-x_\nu}+\right.\right.$$
$$\left.\left.+\frac{H_n'(iut)-(iutx_\nu+1-x_\nu^2)l_\nu(iut)H_n'(iux_\nu)}{(iut-x_\nu)^2}\right\}dt\right].$$

6. Proof of Theorem III. For the determination of $\varrho_\nu(x)$ we need the following

LEMMA 6.1. *We have*

$$\text{(6.1)}\quad \int_0^x e^{\frac{t^2}{2}}H_{2k}(t)\,dt=2\sum_{r=1}^{k}(-12)^{r-1}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{3}{2}-r\right)}e^{\frac{x^2}{2}}H_{2k+1-2r}(x)+$$
$$+(-12)^k\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}}\int_0^x e^{\frac{t^2}{2}}\,dt$$

and

$$\text{(6.2)}\quad \int_0^x e^{\frac{t^2}{2}}H_{2k+1}(t)\,dt=2\sum_{r=1}^{k+1}(-12)^{r-1}\frac{k!}{(k+1-r)!}\left\{e^{\frac{x^2}{2}}H_{2k+2-2r}(x)-H_{2k+2-2r}(0)\right\}.$$

PROOF. The proof follows at once from the formulae

$$\int_0^x e^{\frac{t^2}{2}}H_{2k}(t)\,dt=2e^{\frac{x^2}{2}}H_{2k-1}(x)-6(2k-1)\int_0^x e^{\frac{t^2}{2}}H_{2k-2}(t)\,dt$$

and

$$\int_0^x e^{\frac{t^2}{2}}H_{2k+1}(t)\,dt=2\left\{e^{\frac{x^2}{2}}H_{2k}(x)-H_{2k}(0)\right\}-6\cdot 2k\int_0^x e^{\frac{t^2}{2}}H_{2k-1}(t)\,dt.$$

LEMMA 6.2. *When n is even,*

$$\int_0^x e^{\frac{t^2}{2}} \frac{H_n(t)}{t-x_\nu}\,dt = \frac{2^{n+1}(n-1)!}{H_{n-1}(x_\nu)}\left[\sum_{r=0}^{\frac{n}{2}-1} \frac{H_{2r}(x_\nu)}{2^{2r}(2r)!}\cdot\right.$$

$$(6.3)\quad \cdot\left\{\sum_{i=1}^{r}(-12)^{i-1}\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(r+\frac{3}{2}-i\right)} e^{\frac{x^2}{2}} H_{2r+1-2i}(x) + \frac{(-12)^r\,\Gamma\left(r+\frac{1}{2}\right)}{2\sqrt{\pi}}\int_0^x e^{\frac{t^2}{2}}\,dt\right\} +$$

$$\left. + \sum_{r=0}^{\frac{n}{2}-1} \frac{H_{2r+1}(x_\nu)\,r!}{2^{2r+1}(2r+1)!} \sum_{i=1}^{r+1} \frac{(-12)^{i-1}}{(r+1-i)!}\left\{e^{\frac{x^2}{2}} H_{2r+2-2i}(x) - H_{2r+2-2i}(0)\right\}\right].$$

PROOF. From (3.8) we have

$$\int_0^x e^{\frac{t^2}{2}} \frac{H_n(t)}{t-x_\nu}\,dt = \frac{2^n(n-1)!}{H_{n-1}(x_\nu)} \sum_{r=0}^{n-1} \frac{H_r(x_\nu)}{2^r r!} \int_0^x e^{\frac{t^2}{2}} H_r(t)\,dt.$$

Now breaking the right-hand sum according to even and odd values of r and applying Lemma 6.1, we get the required result.

Determination of $\varrho_\nu(x)$ when n is even. Let

$$(6.4)\qquad \mu_\nu(x) = \frac{H_n(x)e^{-\frac{x^2}{2}}}{2H_n'(x_\nu)}\left[a\int_0^x e^{\frac{t^2}{2}} H_n(t)\,dt + \frac{1}{H_n'(x_\nu)}\int_0^x e^{\frac{t^2}{2}}\frac{H_n(t)}{t-x_\nu}\,dt + b\right].$$

It is easy to see with the help of (3.2) that conditions (5.4) and (5.5) are satisfied for all values of a and b. So

$$\mu_\nu(x) \equiv \varrho_\nu(x).$$

For the determination of a and b we shall make use of the fact that the right-hand side of (6.4) is a polynomial of degree $\leqq 2n-1$, so that the coefficient of

$$\int_0^x e^{\frac{t^2}{2}}\,dt \quad\text{and}\quad e^{-\frac{x^2}{2}}$$

must vanish separately. From these we get a and b as given in (5.13) and (5.14), respectively.

Hence $\varrho_\nu(x)$ is uniquely determined.

7. For the determination of $r_\nu(x)$ we require the following

LEMMA 7.1. *We have*

$$-\int_0^x e^{\frac{1}{2}t^2} \frac{H_n'(t)-(tx_\nu+1-x_\nu^2)l_\nu(t)H_n'(x_\nu)}{(t-x_\nu)^2}\,dt =$$

$$\tag{7.1} = \frac{2^{n+1}(n-1)!}{H_{n-1}(x_\nu)}\Bigg[\sum_{r=0}^{\frac{n}{2}-1}\frac{(n-2r-1)}{2^{2r}(2r)!}H_{2r}(x_\nu)\Bigg\{\sum_{i=1}^{r}(-12)^{i-1}\cdot \frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(r+\frac{3}{2}-i\right)}e^{\frac{x^2}{2}}H_{2r+1-2i}(x)+(-12)^r\frac{\Gamma\left(r+\frac{1}{2}\right)}{2\sqrt{\pi}}\int_0^x e^{\frac{t^2}{2}}dt\Bigg\}+$$

$$+\sum_{r=0}^{\frac{n}{2}-1}\frac{(n-2r-2)H_{2r+1}(x_\nu)}{2^{2r+1}(2r+1)!}\sum_{i=1}^{r+1}(-12)^{i-1}\frac{r!}{(r+1-i)!}\Big\{e^{\frac{x^2}{2}}H_{2r+2-2i}(x)-H_{2r+2-2i}(0)\Big\}\Bigg].$$

PROOF. From (3.8) when $x=t$, we get

$$\frac{H_n(t)}{t-x_\nu}=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-1}\frac{H_r(x_\nu)H_r(t)}{2^r r!}.$$

Differentiating this twice, we have

$$\frac{H_n'(t)}{t-x_\nu}-\frac{H_n(t)}{(t-x_\nu)^2}=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-1}\frac{H_r(x_\nu)H_r'(t)}{2^r r!}$$

and

$$\frac{H_n''(t)}{t-x_\nu}-\frac{2H_n'(t)}{(t-x_\nu)^2}+\frac{2H_n(t)}{(t-x_\nu)^3}=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-1}\frac{H_r(x_\nu)H_r''(t)}{2^r r!}.$$

From these it can easily be seen that

$$-\frac{H_n'(t)-(tx_\nu+1-x_\nu^2)l_\nu(t)H_n'(x_\nu)}{(t-x_\nu)^2}=$$

$$=-\frac{H_n'(t)}{(t-x_\nu)^2}+\frac{x_\nu H_n(t)}{(t-x_\nu)^2}+\frac{H_n(t)}{(t-x_\nu)^3}=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-2}\frac{(n-r-1)}{2^r r!}H_r(x_\nu)H_r(t).$$

Therefore

$$-\int_0^x e^{\frac{t^2}{2}}\frac{H_n'(t)-(tx_\nu+1-x_\nu^2)l_\nu(t)H_n'(x_\nu)}{(t-x_\nu)^2}\,dt=$$

$$=\frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\sum_{r=0}^{n-2}\frac{(n-r-1)}{2^r r!}H_r(x_\nu)\int_0^x e^{\frac{t^2}{2}}H_r(t)\,dt.$$

Now breaking the right-hand sum in even and odd sums and applying Lemma 6.2 we get the required lemma.

Determination of $r_\nu(x)$ when n is even. Let

$$\lambda_\nu(x) = l_\nu^2(x) + \frac{H_n(x)}{H_n'(x_\nu)}\Bigg[Ae^{-\frac{x^2}{2}}\int_0^x e^{\frac{1}{2}t^2} H_n(t)\,dt + \tag{7.2}$$

$$+ e^{-\frac{x^2}{2}} B \int_0^x e^{\frac{t^2}{2}} \frac{H_n(t)}{t-x_\nu}\,dt - \frac{e^{-\frac{1}{2}x^2}}{H_n'(x_\nu)} \int_0^x e^{\frac{t^2}{2}} \frac{H_n'(t)-(Ct+D)l_\nu(t)}{(t-x_\nu)^2}\,dt + Ke^{-\frac{x^2}{2}}\Bigg].$$

Condition (5.2) is obviously satisfied for all values of A, B, C, D and K, i.e.

$$\lambda_\nu(x_j) = \begin{cases} 0 & \text{for} \quad j \neq \nu \\ 1 & \phantom{\text{for}} \quad j = \nu, \end{cases}$$

and for condition (5.3), $j \neq \nu$, it is easy to see with the help of (3.2) and (3.3) that

$$\lambda_\nu''(x_j) = 0.$$

But for $j = \nu$, B is chosen so that

$$\lambda_\nu''(x_\nu) = 0,$$

i.e.

$$2BH_n'(x_\nu) - 2\lim_{x\to x_\nu} \frac{H_n'(x)-(Cx+D)l_\nu(x)}{H_n'(x_\nu)(x-x_\nu)^2} + 2[l_\nu'^2(x_\nu) + l_\nu(x_\nu)l_\nu''(x_\nu)] = 0. \tag{7.3}$$

Now since $\dfrac{H_n'(x)-(Cx+D)l_\nu(x)}{(x-x_\nu)^2}$ is a polynomial, we have

$$H_n'(x_\nu) - (Cx_\nu + D)l_\nu(x_\nu) = 0,$$

$$H_n''(x_\nu) - Cl_\nu(x_\nu) - (Cx_\nu + D)l_\nu'(x_\nu) = 0.$$

From these it is easy to see with the help of (3.4) and (3.5) that

$$C = x_\nu H_n'(x_\nu) \quad \text{and} \quad D = (1-x_\nu^2)H_n'(x_\nu).$$

From (7.2), (3.6), (3.5) and (3.4) we get

$$B = -\frac{2x_\nu^2}{H_n'(x_\nu)}.$$

So $\lambda_\nu(x)$ satisfies both conditions (5.2) and (5.3). Therefore

$$\lambda_\nu(x) \equiv r_\nu(x).$$

For the determination of A and K we shall make use of the fact that the right-hand side of (7. 2) is a polynomial of degree $\leqq 2n-1$ so that

(i) the coefficient of $\int_0^x e^{+\frac{t^2}{2}} dt$ is 0

and

(ii) the coefficient of $e^{-\frac{x^2}{2}}$ is 0.

Now applying (6. 2) and (6. 3) in (7. 2) we get (5. 7) from condition (i). From (ii) we get (5. 11).

Hence $r_\nu(x)$ is uniquely determined as given in (5. 6).

8. Problem of explicit representation. (Case (0, 1, 3).) Given the n distinct zeros

$$+\infty > x_1 > x_2 > \cdots > x_n > -\infty$$

of $H_n(x)$ and arbitrary numbers

$$a_1, a_2, a_3, \ldots, a_n;\ b_1, b_2, b_3, \ldots, b_n;\ c_1, c_2, c_3, \ldots, c_n,$$

we know from Theorem II that there exists a polynomial $R_n(x)$ of degree $\leqq 3n-1$ such that

$$R_n(x_\nu) = a_\nu, \quad R_n'(x_\nu) = b_\nu, \quad R'''(x_\nu) = c_\nu \qquad (\nu = 1, 2, \ldots, n).$$

So for $n = 2k$ we have

$$R_{2k}(x) = \sum_{\nu=1}^{2k} a_\nu u_\nu(x) + \sum_{\nu=1}^{2k} b_\nu r_\nu(x) + \sum_{\nu=1}^{2k} c_\nu w_\nu(x)$$

where $u_\nu(x)$, $r_\nu(x)$ and $w_\nu(x)$ are the fundamental polynomials of the first, second and third kind of (0, 1, 3)-interpolation, belonging to the x_j-points, are polynomials of degree $\leqq 3n-1 = 6k-1$ uniquely determined by the given expressions:

$$(8.\ 1) \qquad u_\nu(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \phantom{\text{for }} j = \nu, \end{cases} \quad u_\nu'(x_j) = u_\nu'''(x_j) = 0 \quad (j = 1, 2, \ldots, n),$$

$$(8.\ 2) \quad r_\nu(x_j) = 0, \quad u_\nu'(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \phantom{\text{for }} j = \nu, \end{cases} \quad r_\nu'''(x_j) = 0 \quad (j = 1, 2, \ldots, n),$$

$$(8.\ 3) \quad w_\nu(x_j) = 0, \quad w_\nu'(x_j) = 0, \quad w_\nu'''(x_j) = \begin{cases} 0 & \text{for } j \neq \nu \\ 1 & \phantom{\text{for }} j = \nu \end{cases} \quad (j = 1, 2, \ldots, n).$$

THEOREM IV. (Case (0, 1, 3).) *For the fundamental polynomials $u_\nu(x)$, $v_\nu(x)$ and $w_\nu(x)$ the following explicit forms can be given:*

$$u_\nu(x)=l_\nu^3(x)+c_\nu v_\nu(x)+\frac{H_n^2(x)}{H_n'^2(x)}\Big\{Ae^{-x^2}\int_0^x e^{t^2}H_n(t)\,dt+ \tag{8.4}$$

$$+Be^{-x^2}\int_0^x e^{t^2}\frac{H_n(t)}{t-x_\nu}dt-\frac{e^{-x^2}}{H_n'(x_\nu)}\int_0^x e^{t^2}\frac{H_n'(t)-(ct^2+dt+a)l_\nu(t)}{(t-x_\nu)^3}dt+ke^{-x^2}\Big\},$$

where c_ν, A, B, c, d, a and k are constants given by

$$c_\nu=-3x_\nu, \tag{8.5}$$

$$A=\frac{(-2)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{6\sqrt{\pi}H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}[(n-2r)H_{2r+1}(x_\nu)+ \tag{8.6}$$

$$+\{-10x_\nu(n-1)+2x_\nu(r+9x_\nu^2)H_{2r}(x_\nu)\}]\frac{\Gamma\left(r+\frac{1}{2}\right)}{(2r)!}(-2)^r,$$

$$B=-\frac{3x_\nu}{H_n'(x_\nu)}[2x_\nu^2+(1-n)], \tag{8.7}$$

$$c=\frac{x_\nu^2+2(1-n)}{3}H_n'(x_\nu), \tag{8.8}$$

$$d=x_\nu\left[1-\frac{2}{3}\{x_\nu^2+2(1-n)\}\right]H_n'(x), \tag{8.9}$$

$$a=\left[1-x_\nu^2\Big\{1-\frac{1}{3}(x_\nu^2+2(1-n))\Big\}\right]H_n'(x) \tag{8.10}$$

and

$$k=-\frac{2^{n-1}(n-1)!}{3H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{(-1)^r r!}{(2r+1)!}(n-2r-1)H_{2r+2}(x_\nu)+ \tag{8.11}$$

$$+x_\nu(2r-10n+11+18x_\nu^2)H_{2r+1}(x_\nu)]\sum_{i=1}^{r+1}\frac{2^{i-1}}{\sqrt{\pi}}\frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)},$$

$$r_\nu(x) = \frac{H_n(x)}{H_n'(x_\nu)}\left[l_\nu^2(x) + H_n(x)e^{-x^2}\left\{A'\int_0^x e^{t^2}H_n(t)\,dt + \right.\right. \tag{8.12}$$
$$\left.\left. + B'\int_0^x e^{t^2}\frac{H_n(t)}{t-x_\nu}\,dt - \frac{1}{H_n'^2(x_\nu)}\int_0^x e^{t^2}\frac{H_n'(t)-(c't+d')l_\nu(t)}{(t-x_\nu)^2}\,dt + k'\right\}\right],$$

where A', B', c', d' *and* k' *are constants given by*

$$A' = -\frac{(-2)^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}H_n'^2(x_\nu)\,H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{n-2r-1-\dfrac{14x_\nu^2+1-n}{3}}{(2r)!}(-2)^r\cdot \tag{8.13}$$
$$\cdot\Gamma\left(r+\frac{1}{2}\right)H_{2r}(x_\nu),$$

$$B' = -\frac{14x_\nu^2+1-n}{3H_n'^2(x_\nu)}, \tag{8.14}$$

$$c' = x_\nu H_n'(x_\nu), \tag{8.15}$$

$$d' = (1-x_\nu^2)H_n'(x_\nu), \tag{8.16}$$

$$k' = \frac{2^{n-1}(n-1)!}{\sqrt{\pi}H_n'^2(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{n-2r-2-\dfrac{14x_\nu^2+1-n}{3}}{(2r+1)!}\cdot \tag{8.17}$$
$$\cdot(-1)^r r!\,H_{2r+1}(x_\nu)\sum_{i=1}^{r+1}2^{i-1}\frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)},$$

$$w_\nu(x) = \frac{H_n^2(x)e^{-x^2}}{6H_n'^2(x_\nu)}\left[c''\int_0^x e^{t^2}H_n(t)\,dt + \frac{1}{H_n'(x_\nu)}\int_0^x e^{t^2}\frac{H_n(t)}{t-x_\nu}\,dt + d''\right], \tag{8.18}$$

where

$$c'' = +\frac{(-2)^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)}{H_n'(x_\nu)H_{n-1}(x_\nu)}\sum_{r=0}^{\frac{n}{2}-1}\frac{(-1)^r H_{2r}(x_\nu)}{2^r r!} \tag{8.19}$$

and

$$(8.20)\qquad d'' = \frac{2^{n-1}(n-1)!}{H_n'(x_\nu)H_{n-1}(x_\nu)} \sum_{r=0}^{\frac{n}{2}-1} \frac{(-1)^r r!}{(2r+1)!} H_{2r+1}(x_\nu) \sum_{i=1}^{r+1} 2^{i-1} \frac{\Gamma\left(r+\frac{3}{2}-i\right)}{\Gamma(r+2-i)}.$$

Equivalent expressions for (8.6), (8.11), (8.13), (8.17), (8.19) and (8.20) are as follows:

$$(8.6a)\qquad A = \frac{1}{(-8)^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)H_n'(x_\nu)} \left[\int_{-\infty}^{\infty} e^{-t^2} \frac{H_n'(it)-(-ct^2+idt+a)l_\nu(it)}{(it-x_\nu)^3} dt + \right.$$
$$\left. + 3x_\nu(2x^\nu+1-n)\int_{-\infty}^{\infty} e^{-t^2}\frac{H_n(it)}{it-x_\nu} dt \right],$$

$$(8.11a)\qquad k = \frac{i}{\sqrt{\pi}H_n'(x_\nu)} \left[\int_0^1 \frac{du}{\sqrt{1-u^2}} 3x_\nu(2x_\nu^2+1-n) \int_{-\infty}^{\infty} te^{-t^2} \frac{H_n(iut)}{(iut-x_\nu)} dt + \right.$$
$$\left. + \int_0^1 \frac{du}{\sqrt{1-u^2}} \int_{-\infty}^{\infty} te^{-t^2} \frac{H_n'(iut)-\{-cu^2t^2+idut+a\}l_\nu(iut)}{(iut-x_\nu)^3} dt \right],$$

$$(8.13a)\qquad A' = \frac{1}{(-8)^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)H_n^2(x_\nu)} \left[\frac{14x_\nu^2+1-n}{3} \int_{-\infty}^{\infty} e^{-t^2}\frac{H_n(it)}{it-x_\nu} dt + \right.$$
$$\left. + \int_{-\infty}^{\infty} e^{-t^2} \frac{H_n'(it)-(itx_\nu+1-x_\nu^2)l_\nu(it)H_n'(x_\nu)}{(it-x_\nu)^2} dt \right],$$

$$(8.17a)\qquad k' = \frac{i}{\sqrt{\pi}H_n'^2(x_\nu)} \left[\frac{14x_\nu+1-n}{3} \int_0^1 \frac{du}{\sqrt{1-u^2}} \int_{-\infty}^{\infty} te^{-t^2}\frac{H_n(iut)}{iut-x_\nu} dt + \right.$$
$$\left. + \int_0^1 \frac{du}{\sqrt{1-u^2}} \int_{-\infty}^{\infty} te^{-t^2} \frac{H_n'(iut)-(iutx_\nu+1-x_\nu^2)l_\nu(iut)H_n'(x_\nu)}{(iut-x_\nu)^2} dt \right],$$

$$c'' = -\frac{1}{(-8)^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)H_n'(x_\nu)}\int_{-\infty}^{\infty} e^{-t^2}\frac{H_n(it)}{it-x_\nu}\,dt, \tag{8.19a}$$

$$d'' = \frac{-i}{\sqrt{\pi}H_n'(x_\nu)}\int_0^1 \frac{du}{\sqrt{1-u^2}}\int_{-\infty}^{\infty} te^{-t^2}\frac{H_n(iut)}{iut-x_\nu}\,dt. \tag{8.20a}$$

The proof of this theorem can be given on the same lines as given in Theorem III.

We shall require the following formulae which can be derived on the same lines as in Lemmas 6.2, 6.3 and 7.1.

When $n=2k$,

$$\int_0^x e^{t^2}H_{2k}(t)\,dt = \sum_{r=1}^{k}(-8)^{r-1}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{3}{2}-r\right)}e^{x^2}H_{2k+1-2r}(x) + (-8)^k\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt. \tag{8.21}$$

When $n=2k+1$,

$$\int_0^x e^{t^2}H_{2k+1}(t)\,dt = \sum_{r=1}^{k}(-8)^{r-1}\frac{k!}{(k+1-r)!}\left[e^{x^2}H_{2k-2r+2}(x) - H_{2k-2r+2}(0)\right]. \tag{8.22}$$

$$\begin{aligned}\int_0^x e^{t^2}\frac{H_n(t)}{t-x_\nu}\,dt = \frac{2^n(n-1)!}{H_{n-1}(x_\nu)}\Bigg[&\sum_{r=0}^{\frac{n}{2}-1}\frac{H_{2r}(x_\nu)}{2^{2r}(2r)!}\left\{\sum_{i=1}^{r}(-8)^{i-1}\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(r+\frac{3}{2}-i\right)}\cdot e^{x^2}H_{2r+1-2i}(x) + (-8)^r\frac{\Gamma\left(r+\frac{1}{2}\right)}{\sqrt{\pi}}\int_0^x e^{t^2}\,dt\right\} + \\ &+\sum_{r=0}^{\frac{n}{2}-1}\frac{H_{2r+1}(x_\nu)}{2^{2r+1}(2r+1)}\sum_{i=1}^{r+1}\frac{(-8)^{i-1}r!}{(r+1-i)!}\left\{e^{x^2}H_{2r+2-2i}(x) - H_{2r+2-2i}(0)\right\}\Bigg].\end{aligned} \tag{8.23}$$

LEMMA 8. 1.

$$\int_0^x \frac{H_n'(t)-(ct^2+dt+a)l_\nu(t)}{(t-x_\nu)^3}e^{t^2}dt =$$

$$= \frac{2^n(n-1)!}{3H_{n-1}(x_\nu)}\left[\sum_{r=0}^{\frac{n}{2}-1}\frac{(n-2r)H_{2r+1}(x_\nu)-x_\nu(n-2r-1)H_{2r}(x_\nu)}{2^{2r}(2r)!}\cdot\right.$$

$$(8.24)\quad \cdot\left\{\sum_{i=1}^{r}(-8)^{i-1}\frac{\Gamma\left(r+\frac{1}{2}\right)}{\Gamma\left(r+\frac{3}{2}-i\right)}e^{x^2}H_{2r+1-2i}(x)+\frac{(-8)^r\Gamma\left(r+\frac{1}{2}\right)}{\sqrt{\pi}}\int_0^x e^{t^2}dt\right\}+$$

$$+\sum_{r=0}^{\frac{n}{2}-1}\frac{(n-2r-1)H_{2r+2}(x_\nu)-x_\nu(n-2r-2)H_{2r+1}(x_\nu)}{2^{2r+1}(2r+1)!}\cdot$$

$$\left.\cdot\sum_{i=1}^{r+1}(-8)^{i-1}\frac{r!}{(r+1-i)!}\left\{e^{x^2}H_{2r-2i+2}(x)-H_{2r-2i+2}(0)\right\}\right]$$

where c, d *and* a *are as given in* (8.8), (8.9) *and* (8.10).

This lemma can be proved easily on the same lines as Lemma 7.1.

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ÜBER KREIS- UND KUGELWOLKEN

Von

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(*Vorgelegt von* G. Hajós)

Der Begriff „Wolke" wurde in einer kürzlich erschienenen Arbeit von L. Fejes Tóth [1] eingeführt. Wir betrachten in der Ebene eine Menge von nicht übereinandergreifenden Einheitskreisen, die zwischen zwei parallelen Geraden liegen. Diese Menge wird Kreiswolke genannt, wenn jede solche Gerade, welche die obigen zwei parallelen Geraden senkrecht schneidet, mindestens einen der Kreise trifft. Man kann es so interpretieren, daß eine Kreiswolke gegen die senkrechten Strahlen eine undurchdringliche Wand bildet. Ganz analog nennen wir im Raum eine Menge von nicht übereinandergreifenden Einheitskugeln eine Kugelwolke, wenn sie zwischen zwei parallelen Ebenen liegt, und wenn jede zu diesen Ebenen senkrechte Gerade mindestens eine der Kugeln trifft.

In der erwähnten Arbeit wurde das Problem aufgeworfen und gelöst, wie groß die minimale Dicke[1] einer Kugelwolke ist. Der Satz von Fejes Tóth lautet folgendermaßen:

Jede Kugelwolke hat eine Dicke $\geq \sqrt{2}+2$, *und Gleichheit gilt nur dann, wenn die Wolke aus zwei quadratischen, einander berührenden Kugelschichten besteht.*

Die minimale Dicke einer Kreiswolke ist offenkundig 2 (Fig. 1).

Fejes Tóth hat meine Aufmerksamkeit auf das Problem der Bestimmung der minimalen Dicke d_k bzw. D_k einer k-fachen Kreis- bzw. Kugelwolke gerichtet, wo eine Wolke k-fach genannt wird, wenn jeder der genannten parallelen Strahlen mindestens k Kreise bzw. Kugeln trifft. (Die früher definierten Wolken sind also in diesem Sinne einfache Wolken.)

Fig. 1

Legen wir k einfache extremale Kreis- bzw. Kugelwolken in geeigneter Weise aufeinander, so erhalten wir eine k-fache Kreiswolke von der Dicke $(k-1)\sqrt{3}+2$ (Fig. 2) bzw. eine k-fache Kugelwolke von der Dicke

[1] Dicke nennen wir — wie gewöhnlich — den Mindestabstand von zwei parallelen Geraden bzw. Ebenen, die die vorliegende ebene bzw. räumliche Menge umfassen.

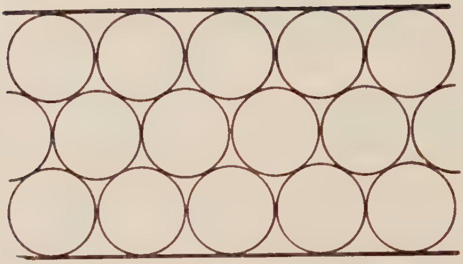

Fig. 2

$(2k-1)\sqrt{2}+2$. Folglich ist

(1) $$d_k \leqq (k-1)\sqrt{3}+2$$

bzw.

(2) $$D_k \leqq (2k-1)\sqrt{2}+2.$$

Beide Wolken liefern einen Teil der dichtesten gitterförmigen Kreis- bzw. Kugellagerung ([2]). Man könnte daher im ersten Augenblick vermuten, daß in (1) und (2) für jedes k das Gleichheitszeichen gilt. Wir werden zeigen, daß dies in der Ebene tatsächlich zutrifft, im Raum aber nicht.

Wir beweisen also folgenden

SATZ. *Bedeutet d_k bzw. D_k die kleinstmögliche Dicke einer k-fachen Wolke von Einheitskreisen bzw. Einheitskugeln, so gilt*

(3) $$d_k = (k-1)\sqrt{3}+2 \quad \textit{für} \quad k>1$$

und

(4) $$D_k < (2k-1)\sqrt{2}+2 \quad \textit{für} \quad k>1.$$

Wir wenden uns zunächst dem Beweis von (3) zu.

Wir betrachten eine beliebige k-fache Kreiswolke in der Ebene, welche in einem Parallelstreifen S von der Breite b enthalten ist. Es seien g_i $(i=1,2)$ die den Streifen begrenzenden Geraden, g_i' eine in S liegende Gerade, die mit g_i den Parallelstreifen S_i von der Breite 1 bestimmt $(i=1,2)$, und schließlich S_3 der von g_1' und g_2' begrenzte mittlere Teilstreifen von S (Fig. 3).

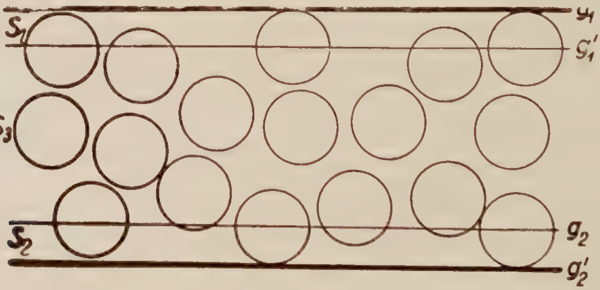

Fig. 3

Ohne Beschränkung der Allgemeinheit können wir voraussetzen, daß die Wolke gesättigt ist, d. h. daß sich zu den Kreisen kein weiterer Kreis hinfügen läßt, so daß eine Wolke von derselben Dicke entsteht.

Wir definieren zunächst die zu einem Kreis gehörige Zelle. Diese besteht aus der Gesamtheit derjenigen Punkte von S, deren Abstand von dem

Mittelpunkt des ausgewählten Kreises kleiner als ihr Abstand von den übrigen Kreismittelpunkten ist. Offenkundig sind die Zellen konvexe Vielecke, die den Streifen S (abgesehen von den Randpunkten der Zellen) schlicht und lückenlos überdecken. Jede Zelle enthält das Innere des zugehörigen Kreises, und wegen der Gesättigtheit der Wolke hat sie bei jeder in S_3 liegenden Ecke einen Winkel $> \frac{\pi}{3}$. Um einen solchen Eckpunkt geschlagener Einheitskreis muß nämlich mindestens den nächsten Kreis der Wolke treffen.

Wir geben jetzt eine andere Zerlegung von S an. Wir verbinden die Mittelpunkte von zwei Kreisen durch eine Strecke, wenn ihre Zellen gemeinsame Randstrecken in S_3 (also im inneren Teilstreifen) haben. Die Verbindungsstrecke und die zugehörige gemeinsame Seite der Zellen sind senkrecht. Diese Strecken bilden ein zusammenhängendes Netz, welches völlig in S_3 liegt (Fig. 4). So ergeben sich zwei äußere, unendliche Bereiche B_1 und B_2, die sich von g_1 bzw. von g_2 bis zum Rande des Netzes erstrecken. Der von S übriggebliebene Teil, der durch das Netz in Vielecke zerlegt wird, sei mit B_3 bezeichnet. Diese Vielecke und die im Inneren von S_3 liegenden Zellenecken sind einander ein-eindeutig zugeordnet. Jedes Vieleck hat soviel Seiten, wieviel Zellen in der zugehörigen Ecke zusammenstoßen. Die Zellenwinkel in dieser Ecke sind komplementäre Winkel des Vielecks. Da aber die Zellenwinkel $> \frac{\pi}{3}$ sind, sind die Winkel des Vielecks $< \frac{2\pi}{3}$. Teilen wir die Vielecke durch einander nicht kreuzende Diagonale weiter, so bekommen wir zum Schluß lauter Dreiecke, deren Eckpunkte in Kreismittelpunkten liegen, deren Seiten $\geqq 2$ und deren Winkel $< \frac{2\pi}{3}$ ausfallen. Daraus folgt, daß in jedem Dreieck die größte Höhe $\geqq \sqrt{3}$ und sämtliche Höhen $\geqq 1$ sind. Folglich is der Inhalt eines Dreiecks stets $\geqq \sqrt{3}$ bzw. der Gesamtinhalt der Wolke innerhalb eines jeden Dreiecks genau $\pi/2$. Hieraus ergibt sich, daß die Dichte[2] der Wolke in B_3 höchstens $\pi/\sqrt{12}$ ist.

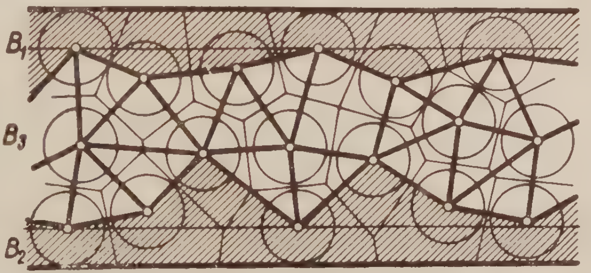

Fig. 4

[2] Unter der Dichte der Wolke in einem im Endlichen liegenden Gebiet G verstehen wir das Inhaltsverhältnis der in G liegenden Kreisteile und von G. Die Dichte in einem unendlichen Gebiet läßt sich dann durch einen Grenzübergang definieren.

Im folgenden Schritt zeigen wir, daß es in B_i einen den Sreifen S_i enthaltenden Teilbereich B_i' gibt, in welchem die Dichte der Wolke den Wert $\pi/4$ nicht übertritt. (Hier und im folgenden bedeutet i stets 1 oder 2.)

Der Bereich B_i ist einerseits von g_i anderseits von einem Streckenzug (vom Rand des Netzes) begrenzt. Diese Strecken verbinden die Mittelpunkte solcher Kreispaare, deren Zellen eine gemeinsame Randstrecke in S_3 und einen gemeinsamen Randpunkt an der Gerade g_i' haben. Es ist leicht einzusehen, daß der Winkel von einer solchen Zellenseite und von g_i' wegen der Gesättigtheit der Wolke größer als $\pi/6$ ist. Deshalb ist der Neigungswinkel zwischen g_i' und einer beliebigen Strecke des Streckenzuges kleiner als $\pi/3$ (Fig. 5).

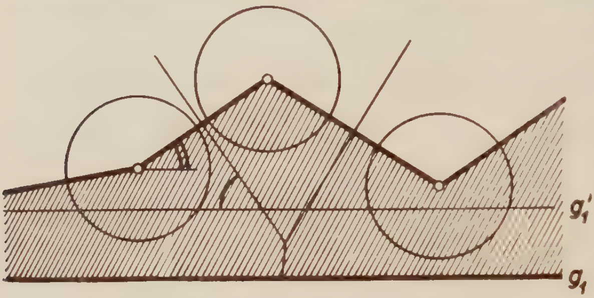

Fig. 5

Die von den Ecken des Streckenzuges auf g_i gefällten Lote zerlegen B_i in rechtwinklige Trapeze (dabei werden als Ecken sämtliche Kreismittelpunkte angesehen, die auf diesem Streckenzug liegen). Zufolge des obigen Resultates ist der Neigungswinkel der Schenkel eines jeden Trapezes kleiner als $\pi/3$. Die Höhe des Trapezes, d. h. der Abstand der parallelen Seiten ist also mindestens 1, deshalb enthält jedes Trapez insgesamt einen halben Kreis aus der Wolke.

Im folgenden wird ein Trapez steil genannt, wenn der Neigungswinkel seiner Schenkel größer als $\pi/6$ ist. Zerlegen wir ein steiles Trapez durch eine Strecke in ein Rechteck und ein rechtwinkliges Dreieck weiter, so enthält sowohl das Rechteck, wie das Dreieck genau einen Viertelkreis der Wolke. Die Dichte der Wolke ist also im Rechteck trivialerweise $\leqq \frac{\pi}{4}$ und im abgeschnittenen rechtwinkligen Dreiecks $\leqq \frac{\pi}{\sqrt{12}}$. Dieses Dreieck ist nämlich die Hälfte eines solchen Dreiecks, welches nur Seiten $\geqq 2$ und Winkel $\leqq \frac{2\pi}{3}$, also einen Inhalt $\geqq \sqrt{3}$ hat. Elementares Rechnen zeigt, daß der Inhalt jedes anderen (nicht steilen) Trapezes wenigstens 2 beträgt, infolgedessen die Dichte der Wolke in diesen Trapezen höchstens $\frac{\pi}{4}$ ist.

Es bedeute B_i' die Gesamtheit der nichtsteilen Trapeze und der Rechtecke der steilen Trapeze, und B_3' die Vereinigung der aus B_1 und B_2 abgeschnit-

tenen rechtwinkligen Dreiecke und der Dreiecke von B_3. Mit Hilfe dieser Bezeichnungen können wir die obigen Ergebnisse folgendermaßen zusammenfassen: Wir haben den Streifen S derart in die Teile B'_j $(j=1,2,3)$ zerlegt, daß die Dichte der Wolke in B'_1 und $B'_2 \leqq \frac{\pi}{4}$ und in $B'_3 \leqq \frac{\pi}{\sqrt{12}}$ ist. Im Hinblick darauf, daß B'_i den Streifen S_i enthält und $\frac{\pi}{\sqrt{12}} > \frac{\pi}{4}$ ist, können wir die folgende Abschätzung für die Dichte D der Wolke in S angeben:

$$D \leqq \frac{1}{b}\left(2\frac{\pi}{4}+(b-2)\frac{\pi}{\sqrt{12}}\right). \tag{5}$$

Projizieren wir die Wolke senkrecht auf g_1, so wirft jeder Kreis eine Schattenstrecke von der Länge 2. Da aber die Schattenstrecken g_1 k-fach überdecken, gehören zu einer Einheitsstrecke von g_1 durchschnittlich mindestens $\frac{k}{2}$ Kreise der Wolke. Daraus ergibt sich für D eine Abschätzung von unten:

$$D \geqq \frac{\frac{k}{2}\pi}{b},$$

woraus sich, mit Rücksicht auf $b \geqq d_k$, (1) und (5), die behauptete Gleichung (3) ergibt.

Um die Gültigkeit der Ungleichung (4) zu zeigen, werden wir eine k-fache Wolke von der Dicke $\left(k+\left[\frac{k-1}{3}\right]\right)\sqrt{3}+2$ konstruieren.[3]

Es seien $\mathbf{a}_1$ und $\mathbf{a}_2$ vom Anfangspunkt O ausgehende Vektoren, die mit O das gleichseitige Dreieck OA_1A_2 von Seitenlänge 2 bestimmen, und $\mathbf{c}$ ein zu $\mathbf{a}_1$ und $\mathbf{a}_2$ senkrechter Vektor von der Länge $\sqrt{3}$. Wir betrachten das von $\mathbf{a}_1$ und $\mathbf{a}_2$ erzeugte Punktgitter und die gitterförmige Kugelschicht G_0, die aus den um diese Gitterpunkte geschlagenen Einheitskugeln besteht. Wir bezeichnen mit G_i die Kugelschicht, welche von G_{i-1} durch die Translation $\frac{\mathbf{a}_i}{2}+\mathbf{c}$ entsteht $(i>0)$, wo $\mathbf{a}_i \equiv \mathbf{a}_1$ bzw. $\mathbf{a}_i \equiv \mathbf{a}_2$, je nachdem i ungerade bzw. gerade ist. Die Kugeln der Schichten greifen nicht übereinander.

Die senkrechte Projektion G'_i der Schicht G_i auf die Ebene des Dreiecks OA_1A_2 liefert eine gitterförmige Lagerung von Einheitskreisen, und es gilt offensichtlich $G'_i = G'_{i-1}+\frac{\mathbf{a}_i}{2}$. Wir greifen aus den Kugelprojektionen die um

[3] Das Symbol $[x]$ bedeutet die größte ganze Zahl, die den Wert x nicht übertrifft.

$O+\frac{\mathbf{a}_1}{2}, O+\frac{\mathbf{a}_1}{2}+\frac{\mathbf{a}_2}{2}$ bzw. $O+\frac{\mathbf{a}_2}{2}$ geschlagenen Einheitskreise K_1, K_2 bzw. K_3 heraus. Es ist leicht einzusehen, daß K_j zum Kreisgitter G'_j gehört ($j=1, 2, 3$), und daß das Dreieck OA_1A_2 durch K_1 und K_2 einfach, durch K_1, K_2 und K_3 zweifach, und durch K_1, K_2, K_3 und G'_4 dreifach überdeckt ist. Folglich liefert wegen der Gitterförmigkeit G_1 und G_2 eine einfache, G_1, G_2 und G_3 eine zweifache, schließlich G_1, G_2, G_3 und G_4 eine dreifache Wolke. Da $G'_i = G'_{i+4}$ ist, ist die Vereinigung der Schichten G_i von $i=1$ bis $i=k+1+\left[\frac{k-1}{3}\right]$ eine k-fache Wolke, welche eine Dicke $\left(k+\left[\frac{k-1}{3}\right]\right)\sqrt{3}+2$ hat. Da aber für $k>1$ $\left(k+\left[\frac{k-1}{3}\right]\right)\sqrt{3}<(2k-1)\sqrt{2}$ ist, ist damit der Beweis von (4) erbracht.

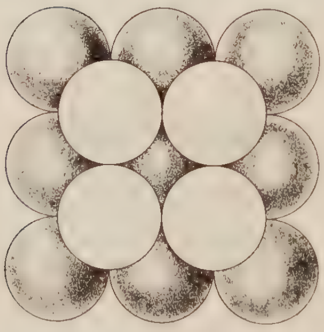

Fig. 6

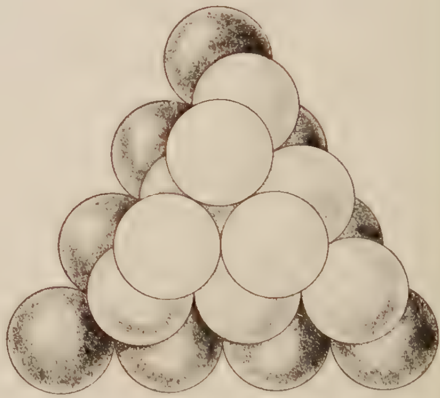

Fig. 7

Fig. 6 bzw. Fig. 7 zeigt die extremale einfache Wolke von FEJES TÓTH bzw. unsere Konstruktion für $k=2$.

(Eingegangen am 5. April 1960.)

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[2] L. FEJES TÓTH, *Lagerungen in der Ebene, auf der Kugel und im Raum* (Berlin—Göttingen—Heidelberg, 1953).

ÜBER DIE ABSOLUTE KONVERGENZ LAKUNAERER TRIGONOMETRISCHER REIHEN

Von

P. SZÜSZ (Budapest)

(*Vorgelegt von* P. TURÁN)

In der vorliegenden Note werden die folgenden beiden Sätze bewiesen:

SATZ 1. *Ist K eine beliebig große positive Zahl, so existiert eine Folge $\{n_k\}$ $(k=1,2,\dots)$ natürlicher Zahlen mit den folgenden Eigenschaften:*

a) *Es ist*

$$\frac{n_{k+1}}{n_k} > K \qquad (k=1,2,\dots).$$

b) *Ist $a_1, a_2, \dots$ eine monoton abnehmende Folge positiver Zahlen, so folgt aus*

$$\sum_{k=1}^{\infty} |a_k \sin \pi n_k x| < \infty \tag{1}$$

für irgendein $x \neq 0, \pm 1, \dots$ die Relation

$$\sum_{k=1}^{\infty} a_k < \infty. \tag{2}$$

SATZ 2. *Es sei $\{n_k\}$ $(k=1,2,\dots)$ eine monoton zunehmende Folge natürlicher Zahlen, für die nur die Relation*

$$\frac{n_{k+1}}{n_k} \to \infty \quad \text{für} \quad k \to \infty \tag{3}$$

besteht. Dann gibt es eine monoton abnehmende Folge $\{a_k\}$ positiver Zahlen, für die zwar $\sum_{k=1}^{\infty} a_k = \infty$ gilt, aber trotzdem gibt es eine Menge der Mächtigkeit des Kontinuums der Zahlen x, für die $\sum_{k=1}^{\infty} |a_k \sin \pi n_k x| < \infty$ gilt.

Satz 1 ist eine Übertragung eines klassischen Satzes von FATOU[1] auf Lückenfolgen. Satz 2 besagt, daß eine dem Fatouschen Satz analoge Behauptung nicht mehr bestehen kann, wenn die Lücken der Folge $\{n_k\}$ „zu groß” sind.

[1] Vgl. z. B. ZYGMUND [1], S. 134.

Beweis des Satzes 1. Ohne Beschränkung der Allgemeinheit sei angenommen, daß K (von a)) eine natürliche Zahl ist ($K \geqq 2$).

Man setze

$$n_1 = 1, \quad n_{k+1} = Kn_k + 1 \qquad (k = 1, 2, \ldots). \tag{4}$$

Nun zeige ich, daß falls $\{n_k\}$ durch (4) definiert ist, so kann für eine beliebige, monoton abnehmende Folge $\{a_k\}$ positiver Zahlen die Relation $\sum_{k=1}^{\infty} |a_k \sin \pi n_k x| < \infty$ mit einem nichtganzen x nur dann stattfinden, wenn $\sum_{k=1}^{\infty} a_k < \infty$ ist. Es darf ohne Beschränkung der Allgemeinheit

$$0 \leqq x < 1$$

angenommen werden; da x voraussetzungsgemäß nichtganz ist, gibt es eine Zahl δ mit $0 < \delta \leqq \frac{1}{2}$ und

$$\delta \leqq x \leqq 1 - \delta. \tag{5}$$

Nun gilt bekanntlich für jedes reelle t [2]

$$|\sin \pi t| \geqq 2 \|t\|; \tag{6}$$

daher genügt es für jede monoton abnehmende Folge $\{a_k\}$ mit $\sum_{k=1}^{\infty} a_k = \infty$ und für jedes nichtganze x die Relation

$$\sum_{k=1}^{\infty} a_k \|n_k x\| = \infty \tag{7}$$

zu beweisen.

Man setze

$$\gamma = \frac{\delta}{8K}, \tag{8}$$

wobei δ durch (5) definiert ist.

Nun zeige ich, daß aus

$$\|n_k x\| < \gamma \tag{9}$$

und

$$\|n_{k+1} x\| < \gamma \tag{10}$$

ein Widerspruch zu (5) folgt. (9) bzw. (10) ist gleichbedeutend mit

$$x = \frac{l_k + \vartheta_k}{n_k} \tag{11}$$

[2] $\|t\|$ bedeutet den Abstand von t von der nächstbenachbarten ganzen Zahl.

bzw.

$$x = \frac{l_{k+1} + \vartheta_{k+1}}{n_{k+1}} \tag{12}$$

mit ganzen l und $|\vartheta_k| < \gamma$, $\vartheta_{k+1}| < \gamma$. Hieraus folgt weiter mit Rücksicht auf (4)

$$|n_k(Kl_k - l_{k+1}) + l_k| < \frac{K+2}{8K} \delta n_k. \tag{13}$$

Aus (5) folgt bei festgelegtem δ für genügend großes n_k

$$\frac{\delta}{2} n_k < l_k < \left(1 - \frac{\delta}{2}\right) n_k. \tag{14}$$

(13) ist gleichbedeutend mit

$$\left| Kl_k - l_{k+1} - \frac{l_k}{n_k} \right| < \frac{\delta}{4}; \tag{13'}$$

da aber K, also auch $Kl_k - l_{k+1}$ ganz ist, ist (13′) mit (14) nicht verträglich. Daher können (für genügend großes n_k) (5), (9) und (10) nicht gleichzeitig bestehen, also falls $0 < x < 1$ ist, muß von irgendeinem k an stets entweder

$$\|n_k x\| > \gamma$$

oder

$$\|n_{k+1} x\| > \gamma$$

sein (oder auch beide). Damit ist (7), also auch unser Satz 1 bewiesen.

Beweis des Satzes 2. Man setze

$$\frac{n_{k+1}}{n_k} = d_k; \tag{15}$$

voraussetzungsgemäß gilt

$$\lim_{k \to \infty} d_k = \infty. \tag{16}$$

Nun sei $\{a_k\}$ eine monoton gegen Null strebende Folge positiver Zahlen, die den Relationen

$$\sum_{k=1}^{\infty} a_k = \infty \tag{17}$$

und

$$\sum_{k=1}^{\infty} \frac{a_k}{d_k} < \infty \tag{18}$$

genügt. Eine solche existiert immer (vgl. Knopp [2], S. 311).

Es gilt für jedes reelle t

$$|\sin \pi t| \leqq \pi \| t \|; \tag{19}$$

daher genügt es zu zeigen, daß für eine Menge der Zahlen x der Mächtigkeit des Kontinuums die Relation

$$\sum_{k=1}^{\infty} a_k \| n_k x \| < \infty \tag{20}$$

besteht.

Die natürliche Zahlenfolge sei in zwei fremde Teilfolgen $k_1, k_2, \ldots$ und $l_1, l_2, \ldots$ eingeteilt. Es bezeichne

$$E(\{k_j\}, \{l_j\}) \tag{21}$$

die Menge der Zahlen x, für die die Relationen

$$\| n_{k_j} x \| \leqq d_{k_j}^{-1} \qquad (j = 1, 2, \ldots) \tag{22}$$

und

$$\| n_{l_j} x \| > d_{l_j}^{-1} \qquad (j = 1, 2, \ldots) \tag{23}$$

gelten.

Kann ich zeigen, daß für jede Einteilung $\{k_j\}$, $\{l_j\}$ die Menge $E(\{k_j\}, \{l_j\})$ nichtleer ist, so bin ich fertig. In diesem Falle wähle ich nämlich eine Teilfolge $l_1, l_2, \ldots$ der natürlichen Zahlenfolge mit $\sum_{j=1}^{\infty} a_{l_j} < \infty$ und verlange, daß für sämtliche k mit $k \neq l_j$ $(j = 1, 2, \ldots)$ die Relation (22) gelten soll.

Für die Zahlen x mit $x \in E(\{k_j\}, \{l_j\})$ ist wegen $\| n_{k_j} x \| \leqq d_{k_j}^{-1}$, wegen $\sum_{j=1}^{\infty} a_{l_j} < \infty$ und wegen (18)

$$\sum_{k=1}^{\infty} a_k \| n_k x \| < \infty. \tag{24}$$

Falls $E(\{k_j\}, \{l_j\})$ bei jeder Einteilung nichtleer ist, so ist die Menge von x mit (24) der Mächtigkeit des Kontinuums, weil ja für l_j bei jedem j die Wahl

$$\| n_{l_j} x \| \leqq d_{l_j}^{-1} \quad \text{oder} \quad \| n_{l_j} x \| > d_{l_j}^{-1}$$

möglich ist.

Daher habe ich nur den folgenden Hilfssatz zu beweisen:

Hilfssatz. *$E(\{k_j\}, \{l_j\})$ ist bei einer beliebigen Zerlegung der natürlichen Zahlenfolge in fremde Teilfolgen $\{k_j\}$ und $\{l_j\}$ nichtleer.*

Beweis. Es bezeichne

$$E_m(\{k_j\}, \{l_j\})$$

die Menge von x, für die die Relationen (22) und (23) gelten, jedoch nicht

notwendigerweise für alle k und l, sondern nur für die mit

$$k_j \leqq m, \quad l_j \leqq m.$$

Offenbar ist

$$E_1 \supseteq E_2 \supseteq \cdots.$$

Nun zeige ich die Existenz einer Folge $I_1, I_2, \ldots$ von Intervallen mit den folgenden Eigenschaften:

a) $I_{n+1} \subset I_n \quad (n = 1, 2, \ldots)$.

b) $I_n \subset E_n \quad (n = 1, 2, \ldots)$.

c) Ist I_n offen, so ist auch die abgeschlossene Hülle von I_{n+1} in I_n enthalten. Da durch das Intervallensystem $I_1, I_2, \ldots$ eine Intervallschachtelung definiert ist, die nicht zu einer nicht in $E(\{k_j\}, \{l_j\})$ enthaltenen Zahl führen kann, wird durch den Nachweis der Existenz einer Intervallenfolge $I_1, I_2, \ldots$ mit a), b) und c) auch unser Hilfssatz bewiesen sein.

Der Beweis der letzten Behauptung wird durch vollständige Induktion geführt.

Für $m = 1$ besteht $E_m(\{k_j\}, \{l_j\})$ aus offenen Intervallen, falls $l_1 = 1$, und aus abgeschlossenen Intervallen, falls $k_1 = 1$, deren Längen gleich $\left(1 - \frac{2}{d_1}\right)\frac{1}{n_1}$ sind, falls $l_1 = 1$ ist, und gleich $\frac{2}{d_1 n_1}$ sind, falls $k_1 = 1$ ist. In $E_1(\{k_j\}, \{l_j\})$ ist also wegen

$$\left(1 - \frac{2}{d_1}\right) > \frac{4}{d_1}, \quad \text{falls} \quad d_1 > 2,$$

stets ein Intervall der Mindestlänge $\frac{1}{n_2}$ enthalten, wenn $k_1 = 1$ ist, und ist ein Intervall der Mindestlänge $\frac{2}{n_2}$ enthalten, wenn $l_1 = 1$ ist (bei der Induktion wird nur so viel ausgenützt werden). Ferner ist dieses Intervall offen für $l_1 = 1$ und abgeschlossen für $k_1 = 1$. Nehmen wir an, dies gilt bis I_m, d. h. I_m ist offen, falls m unter den l vorkommt, und besitzt dann eine Mindestlänge $\frac{2}{n_{m+1}}$, und ist abgeschlossen, falls m unter den k vorkommt, dann besitzt sie eine Mindestlänge $\frac{1}{n_{m+1}}$.

Zunächst sei angenommen, daß m den k angehört. Dann existiert nach der Induktionsvoraussetzung ein abgeschlossenes Intervall I_m von der Mindestlänge $\frac{1}{n_{m+1}}$, die $E_m(\{k_j\}, \{l_j\})$ angehört; dieses enthält (im Inneren oder an

den Endpunkten) mindestens einen Punkt der Gestalt $\frac{r}{n_{m+1}}$; für eine Umgebung von der Mindestlänge $\frac{1}{d_{m+1}n_{m+1}} = \frac{1}{n_{m+2}}$ gilt dann (22), und diese Umgebung bildet wieder ein abgeschlossenes Intervall. Für den übrigen Teil des Intervalles I_m gilt (23). Der kann in höchstens zwei Teilintervalle zerfallen, die Länge von mindestens einem ist also größer als

$$\frac{1}{2}\left(1-\frac{2}{d_{m+1}}\right)\frac{1}{n_{m+1}} > \frac{2}{n_{m+2}},$$

dieses Intervall enthält also ein offenes Teilintervall mit einer größeren Länge als $\frac{2}{n_{m+2}}$. Damit ist die Induktion durchgeführt, falls m den k angehört; der entgegengesetzte Fall kann auf völlig analoge Weise erledigt werden.

Damit ist der Hilfssatz, also auch unser Satz 2 bewiesen.

BUDAPEST, FORSCHUNGSINSTITUT FÜR MATHEMATIK DER
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AN EXTREMAL PROBLEM IN THE THEORY OF INTERPOLATION

By

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1. Let the infinite triangular matrix

$$A=\begin{pmatrix} x_{11} & & & \\ x_{12} & x_{22} & & \\ \vdots & & & \\ x_{1n} & x_{2n} & \cdots & x_{nn} \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

be given, where for $n=1,2,\ldots$ the inequality

$$1\geqq x_{1n}>x_{2n}>\cdots>x_{nn}\geqq -1 \tag{1.1}$$

holds. Putting

$$\omega_n(x,A)=\prod_{j=1}^{n}(x-x_{jn}), \tag{1.2}$$

$$l_{jn}(x,A)=\frac{\omega_n(x,A)}{\omega_n'(x_{jn},A)(x-x_{jn})}, \tag{1.3}$$

the polynomial

$$L_n(x,y_{1n},\ldots,y_{nn},A)=\sum_{j=1}^{n}y_{jn}l_{jn}(x,A), \tag{1.4}$$

the so-called n^{th} Lagrange interpolation polynomial belonging to A, is the only polynomial of degree $\leqq n-1$ having the value y_{jn} at $x=x_{jn}$ for $j=1,2,\ldots,n$. Particularly important is the case when the values y_{jn} are given by

$$y_{jn}=f(x_{jn})\qquad (j=1,2,\ldots,n)$$

where $f(x)$ is a prescribed function continuous in $[-1,+1]$; in this case we shall denote the polynomial in (1.4) more simply by $L_n(x,f,A)$. From the classical investigations of G. FABER[1] and S. BERNSTEIN[2] it follows that *no* matrix A is "effective for the whole class C of functions continuous in

[1] G. FABER [5]. The numbers in brackets refer to the literature quoted at the end of the paper.

[2] S. BERNSTEIN [1].

$[-1, +1]$"; the latter even proved that for every A with (1. 1) there is an $f_0(x) \in C$ and a $-1 \leqq \xi_0 \leqq +1$ such that

$$\overline{\lim_{n \to \infty}} |L_n(\xi_0, f_0, A)| = +\infty,$$

in contrary to everything what was expected since NEWTON.

2. As FEJÉR discovered essentially in 1913, the situation changes completely if instead of the sequence of the Lagrange polynomials $L_n(x, f, A)$ one considers an appropriate special case of the general Hermite interpolation[3] (which HERMITE himself considered only from formal point of view). FEJÉR considered the polynomials $H_n(x, f, A)$ of degree $\leqq 2n-1$ uniquely determined by the requirements

$$\left.\begin{aligned} &H_n(x_{jn}, f, A) = f(x_{jn}), && (2.1)\\ &\left(\frac{dH_n(x, f, A)}{dx}\right)_{x=x_{jn}} = 0 && (2.2) \end{aligned}\right\} \quad (j = 1, 2, \ldots, n).$$

He proved that choosing e. g. for A the matrix P, the n^{th} row of which consists of the roots α_{jn} of the n^{th} Legendre polynomial

$$\{(x^2-1)^n\}^{(n)},$$

one has, whenever $f \in C$, the relation

$$\lim_{n \to \infty} H_n(x, f, P) = f(x)$$

for $-1 < x < +1$, but not necessarily[4] for $x = \pm 1$. Later he proved[5] that choosing as A the matrix T, the n^{th} row of which consists of the roots β_{jn} of the n^{th} Chebyshev polynomial $T_n(x)$ defined by

$$T_n(\cos \vartheta) = \cos n\vartheta, \tag{2.3}$$

the relation

$$\lim_{n \to \infty} H_n(x, f, T) = f(x) \tag{2.4}$$

[3] L. FEJÉR [6].

[4] As it was shown recently by E. EGERVÁRY and P. TURÁN [2] for the sequence of polynomials $H_n^*(x, f)$ of degree $\leqq 2n-3$, defined by

$$H_n^*(\alpha_{j, n-2}, f) = f(\alpha_{j, n-2}), \quad H_n^*(\pm 1, f) = f(\pm 1),$$

$$\left(\frac{dH_n^*(x, f)}{dx}\right)_{x=\alpha_{j, n-2}} = 0 \qquad (j = 1, 2, \ldots, n-2),$$

the relation

$$\lim_{n \to \infty} H_n^*(x, f) = f(x)$$

holds *uniformly for* $[-1, +1]$.

[5] L. FEJÉR [7].

holds uniformly for $[-1, +1]$. Here, generally, $H_n(x, f, A)$ stands for the polynomial of degree $\leqq 2n-1$ defined by

$$\left.\begin{array}{ll} H_n(x_{jn}, f, A) = f(x_{jn}), & (2.5) \\ \left(\dfrac{dH_n(x, f, A)}{dx}\right)_{x=x_{jn}} = y'_{jn} & (2.6) \end{array}\right\} \quad (j=1, 2, \ldots, n)$$

where the real numbers y'_{jn} are subject only to the restriction

$$\lim_{n\to\infty} \max_{j=1,\ldots,n} \frac{|y'_{jn}| \log n}{n} = 0. \tag{2.7}$$

3. The relation (2. 4) is surprising owing to the great arbitrariness of the slopes y'_{jn}. This raises naturally the question that perhaps choosing another matrix A instead of T this arbitrariness of the slopes can be increased. To give a more exact form to this question we remark that, as easy to see,[6] everything depends upon the expression

$$M_n(A) \overset{\text{def}}{=} \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| \tag{3.1}$$

where

$$\mathfrak{h}_{jn}(x, A) = \frac{\omega_n(x, A)^2}{\omega'_n(x_{jn}, A)^2(x-x_{jn})}. \tag{3.2}$$

Hence it is natural to ask for the "optimal" matrix $A = A^*$ (which is not necessarily unique), i. e. for which

$$M_n(A) = \text{minimal} \tag{3.3}$$

for $n = 1, 2, \ldots$. Since, according to FEJÉR,[7] for arbitrarily small $\varepsilon > 0$ for $n > n_0(\varepsilon)$ the inequality

$$M_n(T) < \left(\frac{2}{\pi} + \varepsilon\right) \frac{\log n}{n} \tag{3.4}$$

holds, we certainly have, denoting[8]

$$\min_A M_n(A) = M_n(A^*) \overset{\text{def}}{=} g(n), \tag{3.5}$$

the inequality

$$\overline{\lim_{n\to\infty}} \frac{n}{\log n} g(n) \leqq \frac{2}{\pi}. \tag{3.6}$$

[6] L. FEJÉR [7].
[7] See L. FEJÉR [7] with a slightly different notation.
[8] It is easy to see that for fixed n the minimum exists.

Now we are going to prove

$$\lim_{n\to\infty} \frac{n}{\log n} g(n) \geqq \frac{2}{\pi}, \tag{3.7}$$

i. e.

$$\lim_{n\to\infty} \frac{n}{\log n} g(n) = \frac{2}{\pi}. \tag{3.8}$$

By (3.7) *our extremal problem is at least asymptotically solved and shown that the choice $A=T$ gives essentially the greatest freedom for the choice of the slopes y'_{jn}*. More exactly, we are going to prove the following theorem where c_1 (and later $c_2, c_3, \ldots$) denote positive numerical constants.

THEOREM I. *By whatever choice of the matrix A we have the inequality*

$$(M_n(A) \overset{\text{def}}{=}) \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| \geqq \frac{2}{\pi n} (\log n - c_1 \log\log n).$$

It would be of interest to determine the exact value of $g(n)$, at least for small n's. A proof of the weaker inequality

$$g(n) \geqq c_2 \frac{\log n}{n} \tag{3.9}$$

could have been proved more briefly; we shall, however, omit this version. Probably also the inequality

$$\int_{-1}^{1} \left\{ \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| \right\} dx > c_3 \frac{\log n}{n} \tag{3.10}$$

holds or even the inequality

$$\sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| > c_4 \frac{\log n}{n} \tag{3.11}$$

in $[-1, +1]$ with the exception of a set with measure tending to 0 with $\frac{1}{n}$; we could not prove so far whether or not for all $-1 \leqq a < b \leqq 1$

$$\max_{a\leqq x\leqq b} \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| > \left(\frac{2}{\pi} - \varepsilon\right) \frac{\log n}{n} \tag{3.12}$$

holds for all $n > n_0(\varepsilon, a, b)$ (or even for $n > n_1(\varepsilon)$).

In our theorem the factor $\log\log n$ can perhaps be replaced by 1; a further refinement, enabling to prove that $g(n)$ is a *convex* function of n, seems to be very difficult.

Our method furnishes mutatis mutandis a proof for the inequality

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |l_{jn}(x, A)| \geqq \frac{2}{\pi} \log n - c_5 \log \log n \tag{3.13}$$

for all matrices A; a somewhat weaker inequality was proved in S. BERNSTEIN's paper [1]. The significance of (3.13) is given, of course, by the fact that, in conjunction with the fact that for $n > n_1(\varepsilon)$

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |l_{jn}(x, T)| \leqq \left(\frac{2}{\pi}+\varepsilon\right) \log n,$$

it solves asymptotically the extremal problem to find the minimum of $\max\limits_{-1\leqq x\leqq +1} \sum\limits_{j=1}^{n} |l_{jn}(x, A)|$ when A varies. We shall sketch our proof for (3.13) (Theorem II) and drop the formulation of problems analogous to (3.10), (3.11) and (3.12) with $l_{jn}(x, A)$ instead of $\mathfrak{h}_{jn}(x, A)$.

Since in the proof of our theorem we are always dealing with a large but fixed n, for simplifying the notation we omit n from the indices. Hence for

$$1 \geqq x_1 > x_2 > \cdots > x_n \geqq -1,$$

$$\omega(x) = \prod_{j=1}^{n} (x - x_j), \quad l_j(x) = \frac{\omega(x)}{\omega'(x_j)(x-x_j)}$$

we have to prove that

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} \frac{\omega(x)^2}{\omega'(x_j)^2 |x-x_j|} = \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| = \tag{3.14}$$

$$= \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |x-x_j| \, l_j(x)^2 \geqq \frac{2}{\pi n} (\log n - c_1 \log \log n).$$

4. We shall need two lemmas.

LEMMA I. *If for a $0 < b < \frac{1}{2}$ and $0 < \eta_1 < 1$ and a rational polynomial $J(x)$ of degree n the inequalities*

$$|J(x)| \leqq M \quad \textit{for} \quad -1 \leqq x \leqq +1,$$
$$|J(x)| \leqq \eta_1 M \quad \textit{for} \quad -b \leqq x \leqq +b$$

hold, then for $0 < \eta_2 < \frac{1}{4}$ and

$$-(1-\eta_2)b \leqq x \leqq (1-\eta_2)b$$

the inequality

$$\left|\frac{dJ}{dx}\right| \leqq M \left\{ (1+b^2)\eta_1 n + \frac{4}{\eta_2^2 b^2} \right\}$$

holds.

For the proof of this lemma we may suppose $M=1$, and consider the pure cosine polynomial

$$J(\cos\vartheta)=J_1(\vartheta). \tag{4. 1}$$

We apply the well-known interpolation formula of M. RIESZ [9] which gives

$$\frac{dJ_1}{d\vartheta}=\frac{1}{2n}\sum_{j=1}^{n}J_1(\vartheta+\vartheta_j)\frac{(-1)^{j+1}}{1-\cos\vartheta_j}$$

where

$$\vartheta_j=\frac{(2j-1)\pi}{2n}.$$

Since our hypothesis amounts to

$$|J_1(\vartheta)|\leqq 1 \quad \text{for} \quad 0\leqq\vartheta\leqq\pi,$$

$$|J_1(\vartheta)|\leqq\eta_1 \quad \text{for} \quad \arccos b\leqq\vartheta\leqq\pi-\arccos b,$$

we get for

$$\arccos(1-\eta_2)b\leqq\vartheta\leqq\pi-\arccos(1-\eta_2)b$$

the estimation

$$\left|\frac{dJ_1}{d\vartheta}\right|\leqq\frac{\eta_1}{2n}\sum_{\arccos b\leqq\vartheta+\vartheta_j\leqq\pi-\arccos b}\frac{1}{1-\cos\vartheta_j}+ \tag{4. 2}$$

$$+\frac{\eta_1}{2n}\sum_{\pi+\arccos b\leqq\vartheta+\vartheta_j\leqq 2\pi-\arccos b}\frac{1}{1-\cos\vartheta_j}+\frac{1}{2n}\sum_j{}'\frac{1}{1-\cos\vartheta_j}$$

where the last summation is extended to the ϑ_j's not contained in the previous two. Since

$$\frac{1}{2n}\sum_{j=1}^{n}\frac{1}{1-\cos\vartheta_j}=n,$$

we get

$$\left|\frac{dJ_1}{d\vartheta}\right|\leqq\eta_1 n+\frac{1}{1-\cos(\arccos(1-\eta_2)b-\arccos b)}=\eta_1 n+$$

$$+\frac{1}{1-(1-\eta_2)b^2-\sqrt{1-(1-\eta_2)^2b^2}\cdot\sqrt{1-b^2}}=$$

$$=\eta_1 n+\frac{\{1-(1-\eta_2)b^2\}+\sqrt{1-(1-\eta_2)^2b^2}\cdot\sqrt{1-b^2}}{\{1-(1-\eta_2)b^2\}^2-\{1-(1-\eta_2)^2b^2\}(1-b^2)}<$$

$$<\eta_1 n+\frac{2}{1+(1-\eta_2)^2-2(1-\eta_2)}\frac{1}{b^2}=\eta_1 n+\frac{2}{\eta_2^2}\frac{1}{b^2}.$$

[9] M. RIESZ [9].

Hence for $-(1-\eta_2)b \leqq x \leqq (1-\eta_2)b$

$$\left|\frac{dJ(x)}{dx}\right| = \left|\frac{dJ_1(\vartheta)}{d\vartheta}\right| \frac{1}{\sqrt{1-x^2}} \leqq \left(\eta_1 n + \frac{2}{\eta_2^2 b^2}\right) \frac{1}{\sqrt{1-b^2}} < \eta_1(1+b^2)n + \frac{4}{\eta_2^2 b^2},$$

indeed.

LEMMA II. *Let $J_2(x)$ be a rational polynomial of degree $\leqq m$ which assumes its absolute maximum μ with respect to $[-1, +1]$ at $x = \bar{\xi}$. Then there is an interval I in $[-1, +1]$ of length $\frac{1}{2m^2}$ such that one of its endpoints is $\bar{\xi}$ and in which the inequality*

$$|J_2(x)| \geqq \frac{1}{2}\mu$$

holds.

We choose, namely, as I that one among the intervals

$$\left[\bar{\xi}, \bar{\xi} + \frac{1}{2m^2}\right], \quad \left[\bar{\xi} - \frac{1}{2m^2}, \bar{\xi}\right]$$

which lies in $[-1, +1]$. We may suppose the first. Then using MARKOV's classical theorem[10] we get in I

$$|J_2(x)| = \left|J_2(\bar{\xi}) + \int_{\bar{\xi}}^{x} J_2'(t)\,dt\right| \geqq |J_2(\bar{\xi})| - \int_{\bar{\xi}}^{\bar{\xi}+\frac{1}{2m^2}} \mu m^2\,dt = \mu - \frac{\mu}{2} = \frac{\mu}{2},$$

indeed.

5. We shall employ the following notations. Let

(5. 1) $$M \overset{\text{def}}{=} \max_{-1 \leqq x \leqq +1} |\omega(x)|,$$

and this should be attained here for $x = \xi$, say. We shall consider the intervals

(5. 2) $$d_\nu:\ -\frac{1}{\log n}\left(1 + \frac{1}{\log^2 n}\right)^\nu \leqq x \leqq \frac{1}{\log n}\left(1 + \frac{1}{\log^2 n}\right)^\nu$$

and

(5. 3) $$d_\nu':\ -\frac{1}{\log n}\left(1 + \frac{1}{\log^2 n}\right)^\nu \left(1 - \frac{1}{\log^3 n}\right) \leqq$$
$$\leqq x \leqq \frac{1}{\log n}\left(1 + \frac{1}{\log^2 n}\right)^\nu \left(1 - \frac{1}{\log^3 n}\right)$$

for

(5. 4) $$\nu = 0, 1, \ldots, [\log^2 n] \overset{\text{def}}{=} R.$$

[10] See MARKOV [8].

We shall use $d'_{\nu+1}-d'_\nu$ and $\bar{d}_\nu$ (the complementary of d_ν with respect to $[-1,+1]$) in the usual sense. We shall denote by ξ_ν one of the values x in d_ν with

(5.5) $$|\omega(\xi_\nu)| \overset{\text{def}}{=} \max_{x\in d_\nu} |\omega(x)| \overset{\text{def}}{=} M_\nu.$$

The intervals d_ν are for $n>c_6$ in $[-1,+1]$ and thus

(5.6) $$M_0 \leqq M_1 \leqq \cdots \leqq M_R \leqq M.$$

6. The proof of our Theorem I is split into three cases.

Case I. There is an index $1\leqq k_0\leqq n$ and a $-1\leqq \xi^*\leqq +1$ such that

(6.1) $$\max_{-1\leqq x\leqq +1} |l_{k_0}(x)| = |l_{k_0}(\xi^*)| \geqq n^3.$$

Applying Lemma II to $l_{k_0}(x)$ we obtain the existence of an interval I in $[-1,+1]$ of length $>\dfrac{1}{2n^2}$ such that in I the inequality

(6.2) $$|l_{k_0}(x)| \geqq \frac{1}{2} n^3$$

holds. We choose in I a ξ^{**} as follows. If x_{k_0} is not in I, then let ξ^{**} be the middle-point of I, say; then

(6.3) $$|\xi^{**}-x_{k_0}| \geqq \frac{1}{4n^2}.$$

If x_{k_0} is in I, then ξ^{**} can be chosen in I so that (6.3) holds again. Then we have

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_k(x)| \geqq \sum_{j=1}^{n} |\mathfrak{h}_j(\xi^{**})| \geqq |\mathfrak{h}_{k_0}(\xi^{**})| =$$

$$= |\xi^{**}-x_{k_0}|\, l_{k_0}(\xi^{**})^2 \geqq \frac{1}{4n^2}\,\frac{1}{4}\, n^6 > \frac{2}{\pi}\,\frac{\log n}{n}$$

for $n>c_7$. Hence in this case our theorem is proved and we may suppose in the sequel the inequality

(6.4) $$\max_{-1\leqq x\leqq +1} |l_k(x)| < n^3$$

for $k=1,2,\ldots,n$. This last inequality will be used only in the form that it implies[11] upon the x_j's that writing them in the form

$$x_j = \cos\vartheta_j \qquad (0\leqq \vartheta_j \leqq \pi;\ j=1,2,\ldots,n)$$

[11] See ERDŐS [3]. His proof is an improvement of that contained in ERDŐS—TURÁN [4], esp. p. 548—552.

the ϑ_j's are uniformly distributed in the sense that for $0 \leqq \alpha < \beta < \pi$

$$\left| \sum_{\alpha \leqq \vartheta_j \leqq \beta}^{j} 1 - \frac{\beta - \alpha}{\pi} n \right| < c_8 \log^2 n. \tag{6.5}$$

7. *Case II.* With the notation of **5** we suppose the inequality

$$M_0 < \frac{M}{\log^2 n} \tag{7.1}$$

holds.

We apply Lemma I with

$$J(x) = \omega(x), \quad b = \frac{1}{\log n},$$

$$\eta_1 = \frac{1}{\log^2 n}, \quad \eta_2 = \frac{1}{\log^3 n};$$

the assumption (7. 1) assures the applicability of this lemma. This gives for $x \in d_0'$ the estimation

$$|\omega'(x)| \leqq M \left\{ \left(1 + \frac{1}{\log^2 n}\right) \frac{n}{\log^2 n} + 4 \log^8 n \right\} < M \frac{2n}{\log^2 n}$$

roughly, for $n > c_9$. Hence we obtain

$$\max_{-1 \leqq x \leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| \geqq \sum_{j=1}^{n} |\mathfrak{h}_j(\xi)| \geqq \frac{1}{2} \sum_{j=1}^{n} \frac{\omega(\xi)^2}{\omega'(x_j)^2} \geqq \frac{M^2}{2} \sum_{x_j \in d_0'}^{j} \frac{1}{\omega'(x_j)^2} \geqq \frac{\log^4 n}{8n^2} \sum_{x_j \in d_0'}^{j} 1.$$

Applying (6. 5), the last sum is (roughly) for $n > c_{10}$

$$> \frac{1}{4} \frac{n}{\log n},$$

i. e.

$$\max_{-1 \leqq x \leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| \geqq \frac{\log^3 n}{32n} > \frac{2}{\pi} \frac{\log n}{n}$$

for $n > c_{11}$. Hence also in this case our theorem is proved and in the sequel we may suppose (Case III)

a) the uniformly dense distribution in (6. 5),

b) the inequality

$$M_0 \geqq \frac{M}{\log^2 n}. \tag{7.2}$$

8. *Case III (and the last).* First we assert that there is an index ν_0 with $0 \leqq \nu_0 \leqq [\log^2 n] = R$ and

$$M_{\nu_0+1} \leqq M_{\nu_0} \left(1 + \frac{1}{\log n}\right). \tag{8.1}$$

For if not, then we should have for all these ν's

$$M_{\nu+1} > M_\nu\left(1+\frac{1}{\log n}\right),$$

i. e. from (5. 6), (7. 2) for $n > c_{12}$ by multiplying we get

$$M \geqq M_R > M_0\left(1+\frac{1}{\log n}\right)^R > M_0\sqrt{n} > \frac{\sqrt{n}}{\log^2 n} M > 2M$$

which is false. Hence (8. 1) is true. With this ν_0 we have, with the notations of **5**,

$$\max_{-1\leqq x\leqq +1}\sum_{j=1}^{n}|\mathfrak{h}_j(x)| \geqq \sum_{j=1}^{n}|\mathfrak{h}_j(\xi_{\nu_0})| = \tag{8. 2}$$

$$= \sum_{x_j\in d'_{\nu_0}}{}_j + \sum_{x_j\in d'_{\nu_0+1}-d'_{\nu_0}}{}_j + \sum_{x_j\in \overline{d'_{\nu_0+1}}}{}_j \overset{\text{def}}{=} S_1+S_2+S_3 \geqq S_1+S_2.$$

To obtain a lower bound for S_1 we use Lemma I with $n > c_{13}$ and

$$\eta_1 = \frac{M_{\nu_0}}{M}, \quad \eta_2 = \frac{1}{\log^3 n},$$

$$b = \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\left(> \frac{1}{\log n}\right).$$

This gives for $x_j \in d'_{\nu_0}$ owing to (7. 2) and (5. 6) for $n > c_{14}$

$$|\omega'(x_j)| \leqq M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_0}}{M} n + 4\log^8 n\right\} <$$

$$< M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_0}}{M} n + \left(\frac{M_{\nu_0}}{M}\log^2 n\right) 4\log^8 n\right\} =$$

$$= M_{\nu_0}\left\{\left(1+\frac{25}{\log^2 n}\right) n + 4\log^{10} n\right\} < M_{\nu_0}\left(1+\frac{30}{\log^2 n}\right) n,$$

and hence

$$S_1 = \sum_{x_j\in d'_{\nu_0}}{}_j \frac{M_{\nu_0}^2}{\omega'(x_j)^2|\xi_{\nu_0}-x_j|} > \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^2 n^2}\sum_{x_j\in d'_{\nu_0}}{}_j \frac{1}{|\xi_{\nu_0}-x_j|}. \tag{8. 3}$$

In order to obtain a lower bound for S_2 we apply again Lemma I with

$$\eta_1 = \frac{M_{\nu_0+1}}{M}, \quad \eta_2 = \frac{1}{\log^3 n},$$

$$b = \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0+1}\left(> \frac{1}{\log n}\right).$$

This gives for $x_j \in d'_{\nu_0+1}$, as before,

$$|\omega'(x_j)| \leqq M_{\nu_0+1}\left(1+\frac{30}{\log^2 n}\right)n,$$

i. e. by using (8. 1)

$$S_2 = \sum_{x_j \in d'_{\nu_0+1}-d'_{\nu_0}}{}_j \frac{M_{\nu_0}^2}{\omega'(x_j)^2\,|\xi_{\nu_0}-x_j|} >$$

$$> \frac{M_{\nu_0}^2}{M_{\nu_0+1}^2}\frac{1}{\left(1+\frac{30}{\log^2 n}\right)^2}\frac{1}{n^2}\sum_{x_j \in d'_{\nu_0+1}-d'_{\nu_0}}{}_j \frac{1}{|\xi_{\nu_0}-x_j|} >$$

$$> \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^4}\frac{1}{n^2}\sum_{x_j \in d'_{\nu_0+1}-d'_{\nu_0}}{}_j \frac{1}{|\xi_{\nu_0}-x_j|}.$$

This and (8. 3) give together for $n > c_{15}$

$$S_1+S_2 > \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^4 n^2}\sum_{x_j \in d'_{\nu_0+1}}{}_j \frac{1}{|\xi_{\nu_0}-x_j|}. \tag{8. 4}$$

9. Now we use the full force of the uniform distribution in (6. 5). To do so we write first

$$\xi_{\nu_0} = \cos\Theta_{\nu_0}$$

and have

$$-\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0} \leqq \cos\Theta_{\nu_0} \leqq \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0},$$

i. e.

$$\left|\frac{\pi}{2}-\Theta_{\nu_0}\right| < \arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\right\}; \tag{9. 1}$$

we remark further that the x_j's in (8. 4) are exactly the ϑ_j's with

$$\left|\frac{\pi}{2}-\vartheta_j\right| \leqq \arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0+1}\left(1-\frac{1}{\log^3 n}\right)\right\} \overset{\text{def}}{=} \alpha. \tag{9. 2}$$

Since

$$\frac{1}{|\xi_{\nu_0}-x_j|} = \frac{1}{|\cos\Theta_{\nu_0}-\cos\vartheta_j|} \geqq \frac{1}{|\Theta_{\nu_0}-\vartheta_j|},$$

we have in the remaining Case III

$$\max_{-1\leqq x\leqq+1}\sum_{j=1}^{n}|\mathfrak{h}_j(x)| > \left(1-\frac{30}{\log^2 n}\right)^4\frac{1}{n^2}\sum_{\left|\frac{\pi}{2}-\vartheta_j\right|\leqq\alpha}{}_i \frac{1}{|\Theta_{\nu_0}-\vartheta_j|}. \tag{9. 3}$$

Since from (9. 1) we have

$$\left|\Theta_{\nu_0}-\left(\frac{\pi}{2}\pm\alpha\right)\right|\geqq \arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0+1}\left(1-\frac{1}{\log^3 n}\right)\right\}-$$

$$-\arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\right\}>\arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\left(1+\frac{1}{2\log^2 n}\right)\right\}-$$

$$-\arc\sin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\right\}>\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\frac{1}{2\log^2 n}>\frac{1}{2\log^3 n},$$

the range of summation in (9. 3) is not increased by replacing the original one by

$$|\Theta_{\nu_0}-\vartheta_j|\leqq\frac{1}{2\log^3 n}.$$

Denoting the arcs

$$\Theta_{\nu_0}-(\varkappa+1)\frac{\log^5 n}{n}\leqq\vartheta<\Theta_{\nu_0}-\varkappa\frac{\log^5 n}{n}\qquad\left(\varkappa=0,1,\ldots,\left[\frac{3}{2}\frac{n}{\log^8 n}\right]\right)$$

and

$$\Theta_{\nu_0}+\lambda\frac{\log^5 n}{n}<\vartheta\leqq\Theta_{\nu_0}+(\lambda+1)\frac{\log^5 n}{n}\qquad\left(\lambda=0,1,\ldots,\left[\frac{3}{2}\frac{n}{\log^8 n}\right]\right)$$

by $U_\varkappa$ and V_λ, respectively, (6. 5) results

$$\sum_{\vartheta_j\in V_\lambda}{}^{j}\frac{1}{|\Theta_{\nu_0}-\vartheta_j|}\geqq\frac{n}{\log^5 n}\frac{1}{\lambda+1}\sum_{\vartheta_j\in V_\lambda}{}^{j}1>$$

$$>\frac{n}{\log^5 n}\frac{1}{(\lambda+1)}\left\{\frac{1}{\pi}\log^5 n-c_8\log^2 n\right\}=\frac{n}{\pi}\frac{1}{\lambda+1}\left\{1-\frac{\pi c_8}{\log^3 n}\right\},$$

and similarly for

$$\sum_{\vartheta_j\in U_\varkappa}{}^{j}\frac{1}{|\Theta_{\nu_0}-\vartheta_j|}.$$

Hence from (9. 3) in Case III

$$\max_{-1\leqq x\leqq+1}\sum_{j=1}^{n}|\mathfrak{h}_j(x)|>\left(1-\frac{30}{\log^2 n}\right)^4\frac{2}{\pi n}\left(1-\frac{\pi c_8}{\log^3 n}\right)\sum_{0<\lambda\leqq\left[\frac{n}{\log^8 n}\right]}{}^{\lambda}\frac{1}{\lambda+1}>$$

$$>\frac{2}{\pi}\frac{\log n}{n}-c_{16}\frac{\log\log n}{n}$$

for $n>c_{17}$. Q. e. d.

10. As told we shall sketch the proof of

THEOREM II. *For* $n > c_{18}$ *we have*

$$\max_{-1 \leq x \leq +1} \sum_{\nu=1}^{n} |l_\nu(x)| > \frac{2}{\pi} \log n - c_{19} \log\log n.$$

PROOF. Without loss of generality we may suppose the inequality

$$|l_\nu(x)| \leq \log n \tag{10.1}$$

for $-1 \leq x \leq +1$ and $\nu = 1, 2, \ldots, n$, from which the equidistribution (6.5) follows at once. So we shall have only two cases (keeping the previous notations).

Case I.

$$M_0 < \frac{1}{20 \log^2 n} M. \tag{10.2}$$

We apply Lemma I with

$$J(x) = \omega(x), \quad b = \frac{1}{\log n},$$

$$\gamma_1 = \frac{1}{20 \log^2 n}, \quad \gamma_2 = \frac{1}{\log^3 n};$$

again (10.2) assures the applicability of this lemma. This gives for $x \in d_0'$ as in **7** for $n > c_{20}$

$$|\omega'(x)| < \frac{M}{10} \frac{n}{\log^2 n}$$

and

$$\max_{-1 \leq x \leq +1} \sum_{\nu=1}^{n} |l_\nu(x)| \geq \sum_{\nu=1}^{n} |l_\nu(\xi)| \geq \frac{M}{2} \sum_{x_j \in d_0'} \frac{1}{|\omega'(x_j)|} > 5 \frac{\log^2 n}{n} \sum_{x_j \in d_0'} 1 > \frac{5}{4} \log n$$

using (6.5) roughly.

Case II. We may suppose

$$M_0 \geq \frac{M}{20 \log^2 n}. \tag{10.3}$$

Again we have for $n > c_{21}$ an index ν_1 with $0 \leq \nu_1 \leq R$ and

$$M_{\nu_1+1} \leq M_{\nu_1}\left(1 + \frac{1}{\log n}\right); \tag{10.4}$$

for if not, we should have

$$M > M_0 \sqrt{n} > M \frac{\sqrt{n}}{20 \log^2 n} > 2M$$

which is false. Again

$$\max_{-1\leqq x\leqq +1}\sum_{j=1}^{n}|l_j(x)|\geqq\sum_{j=1}^{n}|l_j(\xi_{\nu_1})|\geqq\sum_{x_j\in d'_{\nu_1}}{}_{j}+\sum_{x_j\in d'_{\nu_1+1}-d'_{\nu_1}}{}_{j}\overset{\text{def}}{=}S'_1+S'_2.$$

To obtain a lower bound for S'_1 we use Lemma I for $n>c_{22}$ with

$$\eta_1=\frac{M_{\nu_1}}{M},\quad \eta_2=\frac{1}{\log^3 n},$$

$$b=\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_1}\left(>\frac{1}{\log n}\right).$$

This gives for $x_j\in d'_{\nu_1}$, using also (10.3), for $n>c_{23}$

$$|\omega'(x_j)|\leqq M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_1}}{M}n+4\log^8 n\right\}<$$

$$<M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_1}}{M}n+\left(\frac{M_{\nu_1}}{M}20\log^2 n\right)4\log^8 n\right\}<M_{\nu_1}\left(1+\frac{30}{\log^2 n}\right)n.$$

The further part of the proof runs exactly after the pattern of Theorem I and can be dropped.

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PROBLEMS AND RESULTS ON THE THEORY OF INTERPOLATION. II

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Let $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$ be n arbitrary points in the interval $(-1, +1)$. $\omega_n(x) = \prod_{i=1}^{n} (x - x_i)$, $l_k(x) = \omega_n(x)/\omega_n'(x_k)(x - x_k)$. It is well known that the sum $\sum_{k=1}^{n} |l_k(x)|$ plays a decisive role in the convergence and divergence properties of the Lagrange interpolation polynomials. FABER [1] proved that $\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)|$ tends to infinity with n, in fact he proved that

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > \frac{1}{12} \log n. \tag{1}$$

Later FEJÉR [2] obtained a very simple proof for (1). The problem of determining the n points for which $\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)|$ is minimal is unsolved up to the present. BERNSTEIN [3] asserts that for every $\varepsilon > 0$, if $n > n_0$,

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > (1-\varepsilon) \frac{2}{\pi} \log n. \tag{2}$$

BERNSTEIN in his important paper proved (2) in full detail for trigonometric interpolation. He states that (2) for interpolation in $(-1, +1)$ is a simple consequence of this result. I was not able to reconstruct the proof. However, we proved with TURÁN [4] that (2) is true, even if the right side is replaced by $\frac{2}{\pi} \log n - c \log\log n$; here and throughout this paper $c, c_1, c_2, \ldots$ will denote positive absolute constants.

The main task of the present paper is the proof of the following

THEOREM 1. *Let* $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$. *Then*

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > \frac{2}{\pi} \log n - c_1.$$

This result can not be improved very much, since it is known that for the roots of the n^{th} Chebyshev polynomial $T_n(x)$

$$\max_{-1\leq x\leq 1} \sum_{k=1}^{n} |l_k(x)| < \frac{2}{\pi} \log n + c_2.$$

In fact, it is known and can be shown by a simple calculation that if $y_1 < y_2 < \cdots < y_n$ are the roots of $T_n(x)$, then

$$\frac{2}{\pi} \log n - c_2 < \max_{y_i < x < y_{i+1}} \sum_{k=1}^{n} |l_k(x)| < \frac{2}{\pi} \log n + c_2.$$

Let $\begin{matrix} & x_1^{(1)} & \\ x_1^{(2)} & & x_2^{(2)} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot \end{matrix}$ be a triangular matrix called point group in the theory of interpolation, $-1 \leqq x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \leqq 1$. BERNSTEIN [3] proved that there exists an x_0 $(-1 < x_0 < 1)$ so that

$$\overline{\lim} \sum_{k=1}^{n} |l_k(x_0)| = \infty.$$

More precisely, he proved that for every fixed $-1 \leqq a < b \leqq 1$

$$\max_{a < x < b} \sum_{k=1}^{n} |l_k(x)| > \left(\frac{1}{4} - \varepsilon\right) \log n \tag{3}$$

for $n > n_0(\varepsilon, a, b)$. I think that in (3) $\frac{1}{4}$ can be replaced by $\frac{2}{\pi}$, but I have not been able to prove this.

In my paper [5] I stated that I can prove that there exists an x_0 so that for infinitely many n

$$\sum_{k=1}^{n} |l_k(x_0)| > \frac{2}{\pi} \log n - c. \tag{4}$$

(4) is quite possibly true, but unfortunately I am very far from being able to prove it.

To prove our Theorem we first need some lemmas.

LEMMA 1. *Let* $\cos \theta_i = y_i$ $(1 \leqq i \leqq n)$ *be the roots of the* n^{th} *Chebyshev polynomial* $T_n(x)$. *Then for every* $-1 \leqq x \leqq 1$ *and* $t > c_3$

$$\frac{1}{n} \sum\nolimits_t \left| \frac{(1-y_i^2)^{\frac{1}{2}}}{x - y_i} \right| > \frac{2}{\pi} \log n - c_4 \log t,$$

where $\sum_t$ *denotes that the summation is extended only over those* y_i*'s for which* $|\theta - \theta_i| > t\pi/n$, $\cos \theta = x$.

The proof of Lemma 1 is by simple computation and is left to the reader.

$\cos\vartheta_0 = x_0$ will denote the point in $(-1, +1)$ where $|\omega_n(x)|$ assumes its absolute maximum. $\bar{I}_t$ will denote the intersection with $(0, \pi)$ of an interval of length $t\pi/n$, one endpoint of which is ϑ_0, I_t will be the interval in $(-1, +1)$ obtained from $\bar{I}_t$ by the mapping $\cos\vartheta = x$. There are two intervals I_t, one to the right, the other to the left of x_0.

LEMMA 2. *Assume that there exists a $t > c_3$ so that for every $t' \geq t$ every interval $I_{t'}$ contains more than $t'\left(1 - \frac{1}{(\log t')^2}\right)$ x_i's. Then*

$$\frac{1}{n}\sum_{i=1}^{n}\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i}\right| > \frac{2}{\pi}\log n - c_5 \log t.$$

The term $(1-x_i^2)^{\frac{1}{2}}$ is really understood to mean $\max\left(|(1-x_i^2)^{\frac{1}{2}}|, \frac{1}{n}\right)$, to save space I will always replace this by $|(1-x_i^2)^{\frac{1}{2}}|$.

Let y_i be such that there are k y's in the interval (x_0, y_i), and let $x_{i'}$ be such that there are k x's in $(x_0, x_{i'})$. Clearly $\theta_i - \theta_0 = \frac{k\pi + O(1)}{n}$ and by our condition on the x's

$$\vartheta_{i'} - \vartheta_0 < \frac{k\pi}{n} + \frac{c_6 k\pi}{n(\log k)^2} + \frac{t\pi}{n} < \frac{k\pi}{n} + \frac{c_7 k\pi}{n(\log k)^2} \tag{5}$$

for $k > t^2$. From (5) we obtain by a simple trigonometrical calculation for $k > t^2$

$$\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i}\right| - \left|\frac{(1-y_i^2)^{\frac{1}{2}}}{y_0 - y_i}\right| > -\frac{c_8}{k(\log k)^2}. \tag{6}$$

Lemma 2 immediately follows from (6) and Lemma 1.

LEMMA 3. *Assume that the x_i's and x_0 have the same properties as in Lemma 2 and the further property that for some $t' > t$ there is an $I_{t'}$ which contains more than t'^3 x_i's. Then if $t > c_3$,*

$$\Sigma = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i}\right| > \frac{2}{\pi}\log n.$$

Let t^* be the greatest t' for which an interval I_{t^*} contains t^{*3} x's. Write

$$\Sigma = \Sigma' + \Sigma_{t^*}$$

where in Σ' $|\vartheta_0 - \vartheta_i| \leq \frac{t^*\pi}{n}$ and in Σ_{t^*} $|\vartheta_i - \vartheta_0| > \frac{t^*\pi}{n}$.

As in the proof of Lemma 2 we can show that

$$\sum\nolimits_{t^*} > \frac{2}{\pi}\log n - c_9 \log t^*. \tag{7}$$

A simple trigonometrical computation shows that for the x_i's in Σ' $\left(\text{here } |\vartheta_i - \vartheta_0| \leqq \frac{t^*\pi}{n} \text{ and by our remark } |(1-x_i^2)^{\frac{1}{2}}| \geqq \frac{1}{n}\right)$

$$\frac{1}{n}\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0 - x_i}\right| > \frac{c_{10}}{t^{*2}}.$$

Thus, since there are at least t^{*3} summands in Σ', we have

$$\Sigma' > ct'. \tag{8}$$

(7) and (8) imply Lemma 3 for sufficiently large $t > c_3$.

LEMMA 4. *Let* $\cos\lambda_0 = x_0$ *be any point in* $(-1, +1)$. *There exists a polynomial* $F_r(x)$ *of degree* r *for which* $F_r(z_0) = 1$ *and*

$$\left|F_r\left[\cos\left(\lambda_0 + s\frac{\pi}{n}\right)\right]\right| < \frac{c_{11}}{|s|}$$

if $\lambda_0 + \frac{s\pi}{n}$ *is in* $(0, \pi)$.

Lemma 4 is well known [6].

LEMMA 5. *Let* $g_m(x)$ *be any polynomial of degree* m, *assume that it assumes its absolute maximum in* $(-1, +1)$ *at* $\cos\lambda_0 = z_0$. *Then if* $\cos\lambda_i = z_i$ *is any root of* $g_m(x)$, *we have*

$$|\lambda_0 - \lambda_i| \geqq \frac{\pi}{2m},$$

equality only holds if $g_m(x) = T_m(x)$.

This is a theorem of M. RIESZ [7].

LEMMA 6. *Assume that the* x_i*'s are such that there is a* $t > c_{12}$ *so that at least one of the intervals* I_t *contains fewer than* $t\left(1 - \frac{1}{(\log t)^2}\right)$ x_i*'s, and that for* $t' \geqq t$ *the intervals* $I_{t'}$ *contain not more than* t'^3 x_i*'s. Then*

$$\max_{x_k \subset I_t} \max_{x \text{ in } J_t} |l_k(x)| > t$$

where by J_t $(J_t \subset I_t)$ *we denote the interval*

$$J_t = \left\}\cos\left(\vartheta_0 + \frac{t\pi}{n(\log t)^3}\right), \cos\left(\vartheta_0 + \frac{t\pi}{n} - \frac{t\pi}{n(\log t)^3}\right)\right\{.$$

Lemma 6 is very far from being best-possible, the conditions could be weakened and the conclusions strengthened, but it will suffice for our purpose in its present form. The proof of Lemma 6 is the most difficult part of the paper [8].

Let $g(x)$ be a polynomial whose roots in I_t coincide with those of $\omega_n(x)=\prod_{i=1}^{n}(x-x_i)$ and outside of J_t they coincide with the roots of the m^{th} Chebyshev polynomial $T_m(x)$, $m=\left[n\left(1-\frac{1}{(\log t)^3}\right)\right]$. By our assumptions the degree of $g(x)$ is less than

$$t-\frac{t}{(\log t)^2}+m-t\left(1-\frac{2}{(\log t)^3}\right)<m \tag{9}$$

for $t>c_{12}$ (i.e. the degree of $g_m(x)$ equals the number of x_i in I_t plus m minus the number of roots of $T_m(x)$ in J_t).

From Lemma 5 and (9) it follows that $g(x)$ must assume its absolute maximum for $(-1,+1)$ in J_t at the point $\cos\lambda_0=z_0$, say.

Denote by $I_t^{(l)}$ $(l=1,2,\ldots)$ the intersection with $(-1,+1)$ of the intervals

$$\left\}\cos\left(\vartheta_0+\frac{2^{l-1}t\pi}{n}\right),\ \cos\left(\vartheta_0+\frac{2^{l}t\pi}{n}\right)\right\{ \tag{10}$$

and

$$\left\}\cos\left(\vartheta_0-\frac{(2^{l}-1)t\pi}{n}\right),\ \cos\left(\vartheta_0-\frac{(2^{l-1}-1)t\pi}{n}\right)\right\{.$$

We now apply Lemma 4 with $r=\left[\frac{n(\log t)^4}{t}\right]$. Since $\cos\lambda_0=z_0$ is in J_t and the distance of the endpoints of $\bar{J}_t$ from the endpoints of $\bar{I}_t$ (in ϑ) is $\frac{t\pi}{n(\log t)^3}$, we obtain from Lemma 4 by a simple computation that for the x's in $I_t^{(l)}$

$$|F_r(x)|<\frac{1}{2^l} \tag{11}$$

for sufficiently large t (i.e. the s in Lemma 4 is for $l=1$ not less than $\log t$ [z_0 is in J_t] and for $l>1$ it is not less than $2^{l-1}\log t$).

Consider now

$$G(x)=Ag(x)\,(F_r(x))^{[t/(\log t)^8]} \tag{12}$$

where A is chosen so that $G(z_0)=1$. The degree of $G(x)$ is not greater than $\left(m=\left[n\left(1-\frac{t}{(\log t)^3}\right)\right]\right)$

$$n-\frac{n}{(\log t)^3}+\frac{t}{(\log t)^8}\frac{n(\log t)^4}{t}<n.$$

Thus by the Lagrange interpolation formula (taken on $x_1, x_2, \ldots, x_n$) we have by (12)

$$1=G(z_0)=\sum_{i=1}^{n} G(x_i)\, l_i(z_0). \tag{13}$$

For the x_i's in I_t $G(x_i)=0$. Thus we can write (13) as

$$1=\sum_{l=1}^{\infty}\sum{}^{(l)} G(x_i)\, l_i(z_0) \tag{14}$$

where in $\Sigma^{(l)}$ the summation is extended over the x_i's in $I_t^{(l)}$. The summation in (14) clearly has to be extended only over a finite number of l's.

Since $|g(z_0)|\geqq|g(x)|$ for $-1\leqq x\leqq 1$ and $F_r(z_0)=1$, we obtain from (11) and (12) that

$$|G(x_i)|<\left(\frac{1}{2^l}\right)^{[t/(\log t)^8]} \quad \text{for the } x_i\text{'s in } I_t^{(l)}. \tag{15}$$

Assume now that our Lemma is false. Then for all $i\not\subset I_t$

$$|l_i(z_0)|\leqq t. \tag{16}$$

Further by the assumptions of our Lemma the number of the x_i's in $I_t^{(l)}$ is not greater than $2^{3l+1}t^3$ (since $I_t^{(l)}$ is contained in the union of the two intervals $I_{2^l t}$). Thus, finally, we obtain from (14), (15) and (16) that

$$1<t^4\sum_{l=1}^{\infty}2^{3l+1}\left(\frac{1}{2^l}\right)^{[t/(\log t)^8]}. \tag{17}$$

The terms of the series (17) drop faster than a geometric series of quotient $\frac{1}{2}$, thus (17) implies

$$1<32t^4\left(\frac{1}{2}\right)^{[t/(\log t)^8]}$$

which is clearly false for $t>c_{12}$. This contradiction proves the Lemma.

Now we are ready to prove our Theorem. In fact, we shall show that if x_0 is the place in $(-1,+1)$ where $\omega_n(x)$ assumes its absolute maximum, then

$$\sum_{k=1}^{n}|l_k(x_0)|>\frac{2}{\pi}\log n-c_1 \tag{18}$$

for sufficiently large c_1. We can clearly assume $\omega_n(x_0)=1$ (replacing $\omega_n(x)$ by $c\omega_n(x)$), and thus by the classical theorem of Bernstein

(19) $$|\omega_n'(x_k)| \leqq \min\left(n^2, \frac{n}{|1-x_k^2|^{\frac{1}{2}}}\right).$$

Thus from (19)

(20) $$\sum_{k=1}^{n} |l_k(x_0)| \geqq \frac{1}{n}\sum_{k=1}^{n}\left|\frac{(1-x_k^2)^{\frac{1}{2}}}{x_0-x_k}\right|.$$

Let the constant c_{12} be sufficiently large. If for every $t>c_{12}$ every I_t contains more than $t\left(1-\frac{1}{(\log t)^2}\right)$ x's, then our Theorem follows from (20) and Lemma 2. Assume next that there exists a $t>c_{12}$ for which I_t contains not more than $t\left(1-\frac{1}{(\log t)^2}\right)$ x's, and let t_0 be the largest such t. Assume first that there exists a $t'\geqq t_0$ for which $I_{t'}$ contains more than t'^3 x's, then our Theorem follows from (20) and Lemma 3. If no such t' exists, consider the largest interval I_{t_0} which contains not more than $t_0\left(1-\frac{1}{(\log t_0)^2}\right)$ x_k's. By Lemma 6 there is an x_i not in I_{t_0} so that for a certain z_0 in J_{t_0}

(21) $$|l_i(z_0)|>t_0.$$

Now since z_0 is in J_{t_0} ($\cos\lambda_0=z_0$, $\cos\vartheta_0=x_0$, $\cos\vartheta_i=x_i$, $x_i\not\subset I_{t_0}$),

(22) $$|\vartheta_i-\vartheta_0|\leqq(\log t_0)^3|\vartheta_i-\lambda_0|.$$

Thus from (22) by a simple computation

(23) $$|x_i-x_0|<(\log t_0)^6|x_i-z_0|.$$

From (23), (21) and $|\omega_n(x_0)|\geqq|\omega_n(z_0)|$ we have

(24) $$|l_i(x_0)|>\frac{t_0}{(\log t_0)^6}.$$

From Lemma 2 we have

(25) $$\frac{1}{n}\sum_{k=1}^{n}{}'\left|\frac{(1-x_k^2)^{\frac{1}{2}}}{x_0-x_k}\right|>\frac{2}{\pi}\log n-c_{13}\log t_0$$

where the dash indicates that $k=i$ is omitted. (25) holds, since a simple computation shows from Lemma 5 that

$$\left|\frac{(1-x_i^2)^{\frac{1}{2}}}{x_0-x_i}\right|<c_{14}n.$$

Thus, finally, from (20), (24) and (25) we have

$$\sum_{k=1}^{n} |l_k(x_0)| > \frac{1}{n} \sum_{k=1}^{n}{}' \left| \frac{(1-x_k^2)^{\frac{1}{2}}}{x_0 - x_k} \right| + \tag{26}$$
$$+ |l_i(x_0)| > \frac{2}{\pi} \log n - c_{13} \log t_0 + \frac{t_0}{(\log t_0)^6} > \frac{2}{\pi} \log n$$

if t is sufficiently large ($t > c_{13}$, say). Thus the proof of Theorem 1 is complete.

It would have been possible to organize the proof differently, since it can be shown that I_t can never contain more than t^3 x_i's. In fact, we have the following

THEOREM 2. *Let* $\omega_n(x) = \prod_{i=1}^{n} (x - x_i)$ *(we do not assume that the* x_i*'s are in* $(-1, +1)$*). Assume that* $\omega_n(x)$ *assumes its absolute maximum in* $(-1, +1)$ *at* $\cos \vartheta_0 = x_0$*. Then every interval* I_t *contains at most* $c_{14} t$ *of the* x_i*'s.*

We do not give the proof of Theorem 2. The best value of c_{14} is not known. Perhaps $c_{14} = 2$.

The problem of determining the points $-1 \leq x_1 < \cdots < x_n \leq 1$ for which

$$\int_{-1}^{+1} \sum_{k=1}^{n} |l_k(x)|\, dx$$

is a minimum is unsolved, and so far as I know has not yet been considered. I believe that to every $\varepsilon > 0$ there exists an n_0 so that for $n > n_0$

$$\int_{-1}^{+1} \sum_{k=1}^{n} |l_k(x)|\, dx > (1-\varepsilon) \int_{-1}^{+1} \sum_{k=1}^{n} |L_k(x)|\, dx \tag{27}$$

where $L_k(x) = \dfrac{T_n(x)}{T_n'(y_k)\,(x - y_k)}$ are the fundamental functions of the Lagrange interpolation taken at the roots $y_1, y_2, \ldots, y_n$ of the n^{th} Chebyshev polynomial. I have not been able to prove (27), but I can prove the following weaker

THEOREM 3. *There exists a constant* c_{15} *so that for every* $-1 \leq x_1 < x_2 < \cdots < x_n \leq 1$ *we have*

$$\int_{-1}^{+1} \sum_{k=1}^{n} |l_k(x)|\, dx > c_{15} \log n. \tag{28}$$

In fact, to every ε *there exists a* δ *so that the number of indices* $1 \leq k \leq n$, *for which*

(29)
$$\int_{-1}^{+1} |l_k(x)|\, dx < \frac{\delta \log n}{n},$$

is less than εn, *and the number of* k's, *for which* $\int_{-1}^{+1} l_k(x)\, dx > \frac{c_{16}}{n}$ *is less than* $c_{17} \frac{n}{\log n}$.

We do not give the proof of Theorem 3, it can be obtained by using the methods of my paper [5].

As far as I know the problem of determining the sequence $-1 \leq x_1 < < x_2 < \cdots < x_n \leq 1$ for which

(30)
$$\int_{-1}^{+1} \sum_{k=1}^{n} l_k^2(x)\, dx$$

is minimal has not been considered. It is possible that the integral (30) is minimal if the x_i's are the roots of the integral of the Legendre polynomial. FEJÉR [9] proved that these are the only points for which

$$\sum_{k=1}^{n} l_k^2(x) \leq 1 \quad \text{for} \quad -1 \leq x \leq 1.$$

THEOREM 4. *To every* ε *there exists an* n_0 *so that for every* $n > n_0$ *the integral* (30) *is greater than* $2-\varepsilon$.

We only outline the idea of the proof. If the projections of the points $x_1, x_2, \ldots, x_n$ on the unit circle are not asymptotically uniformly distributed, then there exists a k so that [10]

(31)
$$\max_{-1 \leq x \leq 1} |l_k(x)| > (1+\delta)^n,$$

and from (31) by Markov's theorem

$$\int_{-1}^{+1} l_k^2(x)\, dx > \frac{(1+\delta)^{2n}}{8n^2} > 2$$

for $n > n_0$. Thus we can assume that the projections of the x_k's on the unit circle are asymptotically uniformly distributed. In this case we obtain our Theorem by showing that

(32)
$$\int_{-1}^{+1} \sum_{k=1}^{n} l_k^2(x)\, dx > (1-\varepsilon) \int_{-1}^{+1} \sum_{k=1}^{n} L_k^2(x)\, dx$$

where $L_k(x) = \frac{P_n(x)}{P_n'(z_k)(x-z_k)}$ $\left(P_n(x) = \prod_{k=1}^{n}(x-z_k)\right)$ is the n^{th} Legendre polynomial. The proof of (32) follows easily from the fact that

$$\int_{-1}^{+1} L_k^2(x)\,dx \leqq \int_{-1}^{+1} f_{n-1}^2(x)\,dx$$

where $f_{n-1}(x)$ is any polynomial of degree $\leqq n-1$ for which $f_{n-1}(z_k) = 1$, and by a simple computation. We suppress the details.

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ON A PROBLEM OF BAER AND A PROBLEM OF WHITEHEAD IN ABELIAN GROUPS

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1. Introduction

The first problem of the title, proposed by BAER [1] in 1936, asks for a characterization of those groups F such that $\mathrm{Ext}(F, T) = 0$ for all torsion groups T. (Call such groups *B-groups.*) The second, proposed by J. H. C. WHITEHEAD in 1952, asks for a characterization of those groups F such that $\mathrm{Ext}(F, Z) = 0$, where Z is the integers. (Call such groups *W-groups.*)[1] Both of these problems have been partially solved: BAER [1] proved that any countable B-group is free; K. STEIN [6][2] proved that any countable W-group is free. (We shall give simple homological algebraic proofs of these theorems. The notation and terminology is that of [3].) Since subgroups of B-groups (W-groups) are again B-groups (W-groups), these groups are $\aleph_1$-free. The simplest example of an $\aleph_1$-free group which is not free is Π, the direct product of countably many copies of Z. Only recently, BAER [2], J. ERDÖS [4], and SĄSIADA [5] independently gave proofs that Π is not a B-group. In this paper we generalize their result by showing that separable B-groups are slender. Further, we show that any W-group is slender and separable. In addition, if all B-groups are separable, then any B-group is a W-group. In the last section we show that certain subgroups of Π are not W-groups.

2. Homological algebra

Let R be a commutative ring with unit. A sequence of R-modules and R-homomorphisms $\cdots \to A_n \to A_{n+1} \to A_{n+2} \to \cdots$ is *exact* in case the image of any homomorphism equals the kernel of the next one. In particular, $0 \to A \xrightarrow{f} B$ exact implies f is a monomorphism, and $B \xrightarrow{g} C \to 0$ exact implies g is an epimorphism. An *extension of C by A* is an exact sequence $0 \to A \to B \to C \to 0$. Thus A may be identified with a submodule of B, and $B/A \approx C$. An equiva-

[1] This problem is of homological interest. It asks whether Z is a universal test group for freedom. The dual question — Is there a universal test group for divisibility? — is an easy exercise; Q/Z is such a group, where Q is the rationals.

[2] I am indebted to R. SWAN for this reference.

lence relation is imposed on extensions of C by A, and one may add two equivalence classes by the "Baer sum." Under this operation, the classes of extensions of C by A form an R-module, which is denoted $\mathrm{Ext}_R(C, A)$ or simply $\mathrm{Ext}(C, A)$. The zero element of this module is the class of the *split sequence:* $0 \to A \to A \oplus C \to C \to 0$. $\mathrm{Ext}(C, A) = 0$ if and only if every extension of C by A is split. If $R = Z$, F is a free group if and only if $\mathrm{Ext}(F, A) = 0$ for any group A; D is a divisible group if and only if $\mathrm{Ext}(C, D) = 0$ for any group C.

There is another R-module which can be assigned to a pair of R-modules C and A: $\mathrm{Hom}_R(C, A)$, the R-homomorphisms of C into A. Any homomorphism f: $A \to B$ induces a homomorphism f^*: $\mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B)$ by $f^*(g) = fg$. Also f induces a homomorphism f_*: $\mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ (note the change in direction) by $f_*(h) = hf$. In particular, if f: $A \to A$ is multiplication by $r \in R$, i. e., $f(a) = ra$, then the induced maps f^* and f_* are also multiplication by r. What we have just said remains true if we replace "Hom" by "Ext" with the exception, of course, that the induced homomorphisms f^* and f_* are defined differently.

Of great importance are the two *induced exact sequences.* Suppose $0 \to A \to B \to C \to 0$ is exact, and M is a fixed module. Then the following sequences are exact:

$$(*) \qquad \begin{aligned} &0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to \\ &\to \mathrm{Ext}(M, A) \to \mathrm{Ext}(M, B) \to \mathrm{Ext}(M, C); \end{aligned}$$

$$(**) \qquad \begin{aligned} &0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to \\ &\to \mathrm{Ext}(C, M) \to \mathrm{Ext}(B, M) \to \mathrm{Ext}(A, M). \end{aligned}$$

(A *connecting homomorphism* to pass from the last Hom to the first Ext can be defined.) If R is a principal ideal domain, one can insert "$\to 0$" at the end of (*) and (**).

We shall also need the following two formulas:

$$\mathrm{Ext}(\Sigma A_\alpha, B) \approx \Pi\, \mathrm{Ext}(A_\alpha, B) \quad \text{and} \quad \mathrm{Ext}(A, \Pi B_\beta) \approx \Pi\, \mathrm{Ext}(A, B_\beta).$$

In particular, Ext commutes with finite direct sums in either variable.

It is assumed the reader is familiar with the requisite abelian group theory.

3. New proofs of known results

We adopt the following notation: $Q =$ the rationals; $C(n) =$ the cyclic group of order n; $C(p^\infty) =$ the torsion divisible group of type p^∞; $\Sigma G_i =$ the direct sum of groups G_i; $\Pi G_i =$ the direct product of the G_i; $I_p^* =$ the p-adic integers. If $x \in G$, $[x]$ is the cyclic subgroup generated by x.

When the meaning is clear from the context, we shall abbreviate ΣG_i by Σ, ΠG_i by Π.

LEMMA 0. *Every subgroup of a B-group (W-group) is a B-group (W-group). Every B-group (W-group) is torsion-free.*

PROOF. Let H be a subgroup of the B-group F. Then exactness of $0 \to H \to F$ induces exactness of $\text{Ext}(F, T) \to \text{Ext}(H, T) \to 0$ for any torsion T. Since $\text{Ext}(F, T) = 0$, $\text{Ext}(H, T) = 0$. (A similar argument works for W-groups, where T is replaced by Z.)

Suppose F is a B-group. If F is not torsion-free, it has a cyclic subgroup $C(n)$, which is also a B-group. But $0 \to C(n) \to C(n^2) \to C(n) \to 0$ is a non-split sequence showing $\text{Ext}(C(n), C(n)) \neq 0$, a contradiction.

Suppose F is a W-group. If F is not torsion-free, it has a cyclic subgroup $C(n)$, which is also a W-group. But $0 \to Z \xrightarrow{f} Z \to C(n) \to 0$ is exact, where f is multiplication by n, and does not split. Therefore $\text{Ext}(C(n), Z) \neq 0$, a contradiction.

LEMMA 1. $\text{Ext}\left(C(p^\infty), \sum_{i=1}^{\infty} C(p^i)\right)$ *is uncountable.*

PROOF. Exactness of $0 \to \Sigma C(p^i) \to \Pi C(p^i) \to \Pi/\Sigma \to 0$ induces exactness of $0 \to \text{Hom}(C(p^\infty), \Pi/\Sigma) \to \text{Ext}(C(p^\infty), \Sigma)$. But $C(p^\infty)$ is a summand of Π/Σ so that $\text{Hom}(C(p^\infty), C(p^\infty))$ is a summand of $\text{Hom}(C(p^\infty), \Pi/\Sigma)$. Since $\text{Hom}(C(p^\infty), C(p^\infty)) \approx I_p^*$, it is uncountable.

LEMMA 2. *If L is torsion-free of rank* 1, *L not cyclic, then there exists a countable torsion group T (depending on L) such that* $\text{Ext}(L, T)$ *is uncountable.*

PROOF. *Case 1.* There exists an infinite set of primes P such that L contains an element x divisible by each $p \in P$. Let $Z = [x]$. Then $L/Z = \sum_{p \in P} A_p$, A_p a non-zero p-primary group. Exactness of $0 \to Z \to L \to L/Z \to 0$ induces exactness of $T \to \text{Ext}(L/Z, T) \to \text{Ext}(L, T) \to 0$, where $T = \sum_{p \in P} C(p)$. Since T is countable, it suffices to prove $\text{Ext}(L/Z, T)$ is uncountable. But $\text{Ext}(L/Z, T) = \text{Ext}(\Sigma A_p, T) \approx \Pi\ \text{Ext}(A_p, T)$ which is uncountable since each $\text{Ext}(A_p, T) \neq 0$; (see proof of Lemma 0).

Case 2. L contains an element x of infinite p-height. If $Z = [x]$, $C(p^\infty)$ is a summand of L/Z. Let $T = \sum_{i=1}^{\infty} C(p^i)$. Exactness of $0 \to Z \to L \to L/Z \to 0$ induces exactness of $T \to \text{Ext}(L/Z, T) \to \text{Ext}(L, T) \to 0$. Thus it suffices to show $\text{Ext}(L/Z, T)$ or its summand $\text{Ext}(C(p^\infty), T)$ is uncountable. This has been done in Lemma 1.

If L does not satisfy Case 1, its characteristic has only finitely many non-zero entries; if L does not satisfy Case 2, its characteristic has no ∞'s as entries. Thus, if L satisfies neither case, L is cyclic.

THEOREM 1. *A countable B-group F is free.*

PROOF. We first assume F has finite rank n; we perform an induction on n. If $n=1$, Lemma 2 gives the desired result. For the general case, let $0 \to H \to F \to L \to 0$ be exact, where L is torsion-free of rank 1. H is free, since its rank is $n-1$. We must show L is cyclic to complete the argument. If L is not cyclic, there exists a countable group T with $\operatorname{Ext}(L, T)$ uncountable. We have exactness of $\operatorname{Hom}(H, T) \to \operatorname{Ext}(L, T) \to \operatorname{Ext}(F, T) = 0$. Since H is free of finite rank, $\operatorname{Hom}(H, T) \approx \sum T$. But now we have a countable group with an uncountable quotient, a contradiction. Hence L is cyclic. PONTRJAGIN's Lemma extends the theorem to arbitrary countable B-groups.

REMARK. For this result, it is only necessary that $\operatorname{Ext}(F, T) = 0$ where $T \approx \sum_p \sum_i C(p^i)$.

LEMMA 3. *A W-group F of rank 1 is cyclic.*

PROOF. Suppose F is not cyclic. Exactness of $0 \to Z \to F \to F/Z \to 0$ induces exactness of $\operatorname{Hom}(F, Z) \to \operatorname{Hom}(Z, Z) \to \operatorname{Ext}(F/Z, Z) \to \operatorname{Ext}(F, Z) = 0$. Since F is not cyclic, $\operatorname{Hom}(F, Z) = 0$, and $Z \approx \operatorname{Ext}(F/Z, Z)$. Since Z is indecomposable, F/Z is p-primary; since F is not cyclic, $F/Z \approx C(p^\infty)$. But if $q \neq p$ is a prime, $\operatorname{Ext}(C(p^\infty), Z)$ is q-divisible, since multiplication by q is an automorphism of $C(p^\infty)$ which induces a similar automorphism of $\operatorname{Ext}(C(p^\infty), Z)$. This contradiction completes the proof.

THEOREM 2. *A countable W-group F is free.*

PROOF. By PONTRJAGIN's Lemma, we may assume F has finite rank n; we perform an induction on n. If $n=1$, we use Lemma 3. Suppose $0 \to H \to F \to L \to 0$ is exact, L torsion-free of rank 1. By induction, H is free. This sequence induces exactness of $\operatorname{Hom}(H, Z) \to \operatorname{Ext}(L, Z) \to \operatorname{Ext}(F, Z) = 0$. Hence $\operatorname{Ext}(L, Z)$ is finitely generated. But since L is torsion-free, $\operatorname{Ext}(L, Z)$ is divisible ([3], p. 135). Therefore $\operatorname{Ext}(L, Z) = 0$ and so L is cyclic, by Lemma 3. Hence F is free.

4. New properties of B-groups and W-groups

At this point, an economy of ideas is required. Both the BAER and WHITEHEAD problems are particular cases of a more general problem. Let $\mathfrak{S}$ be a class of abelian groups; find all groups F such that $\operatorname{Ext}(F, S) = 0$ for

all S in $\mathfrak{S}$. Call such a group an *$\mathfrak{S}$-group*. In BAER's problem, $\mathfrak{S}$ is the class of all torsion groups; in WHITEHEAD's problem, $\mathfrak{S}$ has the unique element Z. Observe that the problem may be further generalized by replacing groups by modules. If G is a group, $|G|$ shall denote its cardinality.

LEMMA 4. (The Density Lemma.) *Let F be a torsion-free group, H a pure subgroup such that F/H is divisible. Suppose there is a countable S in $\mathfrak{S}$ such that* $\operatorname{Ext}(Q, S) \neq 0$. *Then if $2^{|H|} < 2^{|F|}$, F is not an $\mathfrak{S}$-group.*

PROOF. Since $2^{|H|} < 2^{|F|}$, $|H| < |F|$, so that $F/H \approx \sum_{|F|} Q$. Now exactness of $0 \to H \to F \to F/H \to 0$ induces exactness of $\operatorname{Hom}(H, S) \xrightarrow{\delta} \operatorname{Ext}(F/H, S) \to \to \operatorname{Ext}(F, S)$. Since S is countable, $|\operatorname{Hom}(H, S)| \leq \aleph_0^{|H|} = 2^{|H|}$. On the other hand, $\operatorname{Ext}(F/H, S) \approx \operatorname{Ext}(\sum_{|F|} Q, S) \approx \prod_{|F|} \operatorname{Ext}(Q, S)$. Since $\operatorname{Ext}(Q, S) \neq 0$, $|\operatorname{Ext}(F/H, S)| \geq 2^{|F|}$. Thus δ cannot be an epimorphism and so $\operatorname{Ext}(F, S) \neq 0$. Thus F is not an $\mathfrak{S}$-group.

We make two remarks here. Suppose we were considering the more general problem of determining $\mathfrak{S}$-modules, where *module* means R-module, R a principal ideal domain. Lemma 4 is still true if we replace Q by the quotient field of R, and if we further assume this quotient field is countable. In particular, the lemma is true if $R = I_p$, the p-adic rationals.

Let us return to groups. A group F is *Hausdorff* in case $\bigcap_n n!F = 0$. If F is Hausdorff, the subgroups $n!F$ define a metric topology on F, *the n-adic topology*, which makes F a topological group. A pure subgroup H of F is a subspace (i. e., the n-adic topology on H is the same topology as that induced on H in virtue of its being contained in F). Also a subgroup H is dense in F if and only if F/H is divisible. Thus we may paraphrase Lemma 4 in saying that if F is an $\mathfrak{S}$-group, no subgroup of smaller cardinality can be dense in F.[3] But even more is true. Since any subgroup of an $\mathfrak{S}$-group is again an $\mathfrak{S}$-group, we know that the closure $\bar{H}$ of a subgroup H has the same cardinality as H.[4]

DEFINITION. Let Z_i be an infinite cyclic group with generator e_i $(i = 1, 2, \ldots)$, and let $\Pi = \prod Z_i$. A group F is *slender* in case $f(e_i) = 0$ for almost all i, f any homomorphism from Π to F.

ŁOS [5] has shown that any free group is slender. Since Π is Hausdorff, we consider it topologized. Let Σ denote the subgroup $\sum Z_i$, and let $\bar{\Sigma}$ be the closure of Σ in Π; $\bar{\Sigma}$ is a subgroup. I conjecture that slenderness is really a reflection of the topological relation between Σ and $\bar{\Sigma}$; in all slen-

[3] We assume $|H|$ and $|F|$ are such that $|H| < |F|$ and $2^{|H|} < 2^{|F|}$.

[4] Unless $|H| < |\bar{H}|$ and $2^{|H|} = 2^{|\bar{H}|}$.

derness arguments one should replace Π by $\bar{\Sigma}$. The following proof (essentially due to SĄSIADA) illustrates this conjecture.

LEMMA 5. *$\bar{\Sigma}$ has no direct summand isomorphic to Σ.*

PROOF. Suppose $\bar{\Sigma} = A \oplus B$, where $A \approx \Sigma$; let f: $\bar{\Sigma} \to A$ be the projection. Suppose $f(\Sigma)$ has infinite rank in A. We may assume $f(e_i) \neq 0$ for all i. Since A is reduced and torsion-free, there exists a sequence of positive integers m_i such that $f(m_i! e_i) \notin m_{i+1}! A$. Set $\mathbf{X} =$ all $x \in \Pi$ such that $(x_i) = 0$ or $\pm m_i!$ ($(x)_i =$ the i^{th} co-ordinate of x); set $\mathbf{Y} =$ all $x \in \Pi$ such that $(x)_i = 0$ or $m_i!$. Note that $\mathbf{Y} \subset \mathbf{X} \subset \bar{\Sigma}$. Further, if $x, y \in \mathbf{Y}$, then $x - y \in \mathbf{X}$. Suppose $x \neq 0$, $x \in \mathbf{X} \cap \ker f$; let the first non-zero co-ordinate of x be the j^{th}. Then $0 = f(x) = f(m_j! e_j) + f(\{0, \ldots, 0, \pm m_{j+1}! e_{j+1}, \ldots\})$, so that $f(m_j! e_j) \in m_{j+1}! A$, a contradiction. Now $\mathbf{Y}$ is uncountable while A is countable. Hence there exist x and $y \in \mathbf{Y}$, $x \neq y$, such that $f(x) = f(y)$. Thus $f(x - y) = 0$ contradicting $\mathbf{X} \cap \ker f = 0$. We must conclude that $f(\Sigma)$ has finite rank in A. But $f(\Sigma)$ is dense in $f(\bar{\Sigma}) = A$, so that A has finite rank, another contradiction.

DEFINITION. A torsion-free group G is *separable* in case any pure subgroup of finite rank is a direct summand of G.[5]

LEMMA 6. *Let $\mathfrak{S}$ be a class of groups containing a countable S such that* $\operatorname{Ext}(Q, S) \neq 0$. *Then a separable $\aleph_1$-free $\mathfrak{S}$-group F is slender.*

PROOF. Suppose f: $\Pi \to F$ with $f(e_i) \neq 0$ for infinitely many i; let H be the pure subgroup generated by $f(\Sigma)$. If H has finite rank, it is a direct summand of F, by separability. Let π: $F \to H$ be the projection. Then πf: $\Pi \to H$ has $\pi f(e_i) \neq 0$ for infinitely many i, contradicting the fact that H is slender (H is free). Hence $f(\Sigma)$ has infinite rank. Now $f(\Sigma)$ is pure and dense in $f(\bar{\Sigma})$; also $f(\bar{\Sigma})$ is an $\mathfrak{S}$-group, so that we may apply the density lemma. Thus countability of $f(\Sigma)$ implies countability of $f(\bar{\Sigma})$. Since $f(\bar{\Sigma}) \subset f(\bar{\Sigma})$ and F is $\aleph_1$-free, $f(\bar{\Sigma})$ is free of countable rank, i. e., $f(\bar{\Sigma}) \approx \Sigma$. But then $\bar{\Sigma}$ has a summand isomorphic to Σ, a contradiction.

THEOREM 3. *A separable B-group is slender.*

PROOF. An immediate consequence of Theorem 1 and Lemma 6.

We now turn our attention to W-groups.

LEMMA 7. *Let F be a W-group with pure subgroup H of finite rank. Then F/H is a W-group.*

PROOF. Exactness of $0 \to H \to F \to F/H \to 0$ induces exactness of $\operatorname{Hom}(H, Z) \to \operatorname{Ext}(F/H, Z) \to \operatorname{Ext}(F, Z) = 0$. Since H has finite rank, it is

[5] This notion of separability is not the usual one, but the two notions coincide in the case of homogeneous groups, and hence in the present problem.

free. Hence $\operatorname{Hom}(H, Z)$ is free of finite rank, so that $\operatorname{Ext}(F/H, Z)$ is finitely generated. Since F/H is torsion-free, $\operatorname{Ext}(F/H, Z)$ is divisible. Therefore $\operatorname{Ext}(F/H, Z) = 0$.

COROLLARY 1. *Any W-group F is separable.*

PROOF. Let H be a pure subgroup of finite rank in F; H is free. Hence $\operatorname{Ext}(F/H, H) \approx \Sigma \operatorname{Ext}(F/H, Z) = 0$, by Lemma 7. Hence $\operatorname{Ext}(F/H, H) = 0$, i. e., H is a direct summand of F.

THEOREM 4. *Any W-group is slender.*

THEOREM 5. *Any W-group F can be imbedded as a pure subgroup in a direct product of Z's.*

PROOF. By Corollary 1, for each $x \in F$, there is a $\delta_x \colon F \to Z$ such that $\delta_x(x) = h(x)$, where $h(x)$ is the height of x. Define $D\colon F \to \prod_{x \in F} Z$ by $D(y) = \{\delta_x(y)\}$. D is a monomorphism, by our initial remark. $D(F)$ is a pure subgroup, since: (1) the height of an element in ΠZ is the minimum of the heights of its co-ordinates; (2) any homomorphism, e. g., δ_x, cannot lower heights.

We have observed that B-groups and W-groups share many properties. It is a plausible conjecture that these two classes of groups are identical. The following two theorems shed some light on this conjecture:

THEOREM 6. *Suppose every B-group is separable. Then any B-group F is a W-group.*

PROOF. Let H be a pure subgroup of F of finite rank. Since F is separable, F/H is a B-group (being a summand of F). If (*) $0 \to H \to G \to F/H \to 0$ is exact, then $\operatorname{Ext}(F/H, T) \to \operatorname{Ext}(G, T) \to \operatorname{Ext}(H, T)$ is exact for any torsion group T. Hence G is a B-group. Since F/H is torsion-free, H is a pure subgroup of G. Therefore H is a summand of G, because G is separable. Thus the sequence (*) always splits, i. e., $\operatorname{Ext}(F/H, H) = 0$. Since H is free of finite rank, $\operatorname{Ext}(F/H, Z) = 0$. But $F \approx H \oplus F/H$. Therefore $\operatorname{Ext}(F, Z) \approx \operatorname{Ext}(H \oplus F/H, Z) \approx \operatorname{Ext}(H, Z) \oplus \operatorname{Ext}(F/H, Z) = 0$. Hence F is a W-group.

I cannot prove that every B-group is separable, but I can prove one result in this direction.

THEOREM 7. *Let F be a B-group, H a pure subgroup of finite rank. Then F/H is* $\aleph_1$*-free.*

We first prove

LEMMA 8. *Let L be a torsion-free group of finite rank which is not free. Then there exists a countable torsion group T such that* $\operatorname{Ext}(L, T)$ *is uncountable.*

(We remark that this lemma gives another proof of Theorem 1.)

PROOF. Assume rank $L = n$. If $n = 1$, this is Lemma 2. Let $0 \to M \to$ $\to L \to L/M \to 0$ be exact, where M is pure of rank $n-1$. This induces exactness of $\mathrm{Hom}(M, T) \to \mathrm{Ext}(L/M, T) \to \mathrm{Ext}(L, T) \to \mathrm{Ext}(M, T) \to 0$, for any torsion T. If M is not free, choose a countable T, by induction, such that $\mathrm{Ext}(M, T)$ is uncountable; then $\mathrm{Ext}(L, T)$ is uncountable. If M is free, then L/M is not free, so that we can choose a countable torsion T so that $\mathrm{Ext}(L/M, T)$ is uncountable. But $\mathrm{Hom}(M, T) \approx \Sigma T$ is countable so that $\mathrm{Ext}(L, T)$ is uncountable.

We return to the proof of Theorem 7.

Exactness of $0 \to H \to F \to F/H \to 0$ induces exactness of $\mathrm{Hom}(H, T) \to$ $\to \mathrm{Ext}(F/H, T) \to \mathrm{Ext}(F, T) = 0$, T any torsion group. Since H is free of finite rank, $\mathrm{Hom}(H, T) \approx \Sigma T$. Hence $|\mathrm{Ext}(F/H, T)| \leqq |T|$. Let G be a subgroup of F/H of finite rank. Exactness of $0 \to G \to F/H$ induces exactness of $\mathrm{Ext}(F/H, T) \to \mathrm{Ext}(G, T) \to 0$. By Lemma 8, if G is not free, we can choose a countable T such that Ext (G, T) is uncountable, contradicting the inequality above. Hence G must be free. PONTRJAGIN's Lemma completes the proof.

The following questions remain open: If F is a B-group, is F separable? An easier question is: if F is a B-group, H a pure subgroup of finite rank, is F/H a B-group? (An affirmative answer to this question would imply Theorem 7.) Which pure subgroups of a product of copies of Z are slender?

5. Further investigations

We have seen that any W-group can be imbedded as a pure subgroup in a direct product of Z's. Thus one approach to solving WHITEHEAD's problem is to eliminate all non-free subgroups (assuming that the conjecture — all W-groups are free — is correct). I propose the following plan of attack. Let us find a family of subgroups $\{A_i\}$ with two properties: 1. each A_i is not a W-group; 2. if a subgroup S of a direct product of Z's is not free, then it contains a copy of some A_i. Since any subgroup of a W-group is a W-group, such a method could solve the problem.

Let us only look at Π and its subgroups, where Π is a direct product of countably many Z's. As candidates for the A_i, I suggest the following subgroups. G is of *type* $n!$ if it is isomorphic to π^{-1} (the divisible subgroup of Π/Σ), where $\pi\colon \Pi \to \Pi/\Sigma$ is the natural map. G is of type p^ω if it is isomorphic to π^{-1} (the p-divisible subgroup of Π/Σ). The following question remains open: If a pure subgroup S of Π contains no subgroup of type $n!$ or of type p^ω, is S free?

LEMMA 9. *The group G of type $n!$ is not a W-group.*

Proof. G is uncountable with pure dense subgroup Σ. Since Σ is countable, we apply the density lemma.

In order to show groups of type p^n are not W-groups, we must examine I_p-modules.

Lemma 10. *For any group G,* $\operatorname{Ext}_Z(I_p \otimes G, I_p) \approx \operatorname{Ext}_{I_p}(I_p \otimes G, I_p)$.

Proof. Observe that $\operatorname{Hom}_Z(I_p \otimes G, I_p) = \operatorname{Hom}_{I_p}(I_p \otimes G, I_p)$. Exactness of $0 \to I_p \to Q \to C(p^\infty) \to 0$ induces exactness of the rows of the commutative diagram:

$$\begin{array}{ccccccccc}
\operatorname{Hom}_Z(I_p \otimes G, I_p) & \to & \operatorname{Hom}_Z(I_p \otimes G, Q) & \to & \operatorname{Hom}_Z(I_p \otimes G, C(p^\infty)) & \to & \operatorname{Ext}_Z(I_p \otimes G, I_p) & \to & 0, \\
\| & & \| & & | & & & & \\
\operatorname{Hom}_{I_p}(I_p \otimes G, I_p) & \to & \operatorname{Hom}_{I_p}(I_p \otimes G, Q) & \to & \operatorname{Hom}_{I_p}(I_p \otimes G, C(p^\infty)) & \to & \operatorname{Ext}_{I_p}(I_p \otimes G, I_p) & \to & 0.
\end{array}$$

Hence the cokernels are isomorphic.

Lemma 11. *If* $\operatorname{Ext}_Z(G, Z) = 0$, *then* $\operatorname{Ext}_{I_p}(I_p \otimes G, I_p) = 0$.

Proof. We shall show that $\operatorname{Ext}_Z(I_p \otimes G, I_p) = 0$ and the result will follow from Lemma 10. All Ext's appearing in this proof shall be Ext_Z's.

Exactness of $0 \to Z \to I_p \to I_p/Z \to 0$ induces exactness of $\operatorname{Ext}(G, Z) \to$ $\to \operatorname{Ext}(G, I_p) \to \operatorname{Ext}(G, I_p/Z)$. But $I_p/Z \approx \sum_{q \neq p}' C(q^\infty)$ which is divisible; hence $\operatorname{Ext}(G, I_p/Z) = 0$. Since $\operatorname{Ext}(G, Z)$ is also 0, we have $\operatorname{Ext}(G, I_p) = 0$.

Exactness of $0 \to G \to I_p \otimes G \to (I_p/Z) \otimes G \to 0$ (G is torsion-free) induces exactness of $\operatorname{Ext}(I_p/Z \otimes G, I_p) \to \operatorname{Ext}(I_p \otimes G, I_p) \to \operatorname{Ext}(G, I_p) = 0$. Now $\operatorname{Ext}(I_p/Z \otimes G, I_p) \approx \operatorname{Ext}(\sum_{q \neq p}' C(q^\infty) \otimes G, I_p) \approx \prod_{q \neq p} \operatorname{Ext}(C(q^\infty) \otimes G, I_p)$. Exactness of $0 \to I_p \to Q \to C(p^\infty) \to 0$ induces exactness of $0 = \operatorname{Hom}(C(q^\infty) \otimes G, C(p^\infty)) \to$ $\to \operatorname{Ext}(C(q^\infty) \otimes G, I_p) \to \operatorname{Ext}(C(q^\infty) \otimes G, Q) = 0$, since $C(q^\infty) \otimes G$ is q-primary. Hence $\operatorname{Ext}(C(q^\infty) \otimes G, I_p) = 0$, $\operatorname{Ext}(I_p/Z \otimes G, I_p) = 0$, and finally $\operatorname{Ext}(I_p \otimes G, I_p) = 0$.

Corollary 2. *If G is a W-group, $I_p \otimes G$ is a W-module (over the ring I_p).*

Lemma 12. *A group G of type p^n is not a W-group.*

Proof. Otherwise $I_p \otimes G$ would be a W-module, and so would satisfy the module version of the density lemma. But $I_p \otimes G$ is uncountable, while $I_p \otimes \Sigma$ is a pure dense submodule which is countable. This contradiction completes the proof.

Note (added 15 October 1960). Ti Yen succeeded in solving a problem raised at the end of **4.**

Theorem. *Let F be a B-group with pure subgroup H of finite rank. Then F/H is a B-group.*

Exactness of $0 \to H \to F \to F/H \to 0$ induces exactness of $\operatorname{Hom}(F, T) \xrightarrow{\alpha} \operatorname{Hom}(H, T) \to \operatorname{Ext}(F/H, T) \to \operatorname{Ext}(F, T) = 0$ where T is any torsion group. In order to show $\operatorname{Ext}(F/H, T) = 0$, it suffices to prove α is an epimorphism. Let $f\colon H \to T$. Since H is free of finite rank, $f(H)$ is finite. Replacing T by $f(H)$ in the above exact sequence yields $\operatorname{Hom}(F, f(H)) \to \operatorname{Hom}(H, f(H)) \to \to \operatorname{Ext}(F/H, f(H)) = 0$. Hence f can be extended over F to a map into $f(H)$; a fortiori, f can be extended to a map from F to T. Hence α is an epimorphism, $\operatorname{Ext}(F/H, T) = 0$, and F/H is a B-group.

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NOTE ON FULLY ORDERED SEMIGROUPS

By

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(*Presented by* L. RÉDEI)

Dedicated to P. TURÁN *on his 50th birthday*

By a *fully ordered* (briefly: f. o.) *semigroup*[1] S is meant a semigroup which is at the same time a f. o. set under an ordering relation $\leq$ such that $a \leq b$ $(a, b \in S)$ implies $ac \leq bc$ and $ca \leq cb$ for all $c \in S$. (If S is cancellative, then this is equivalent to the fact that $a < b$ implies $ac < bc$ and $ca < cb$, but we do not assume the cancellation laws.) We say that S is *positively ordered* if $ab \geq a$ and $ab \geq b$ for all $a, b \in S$, and *naturally ordered* if it is positively ordered and $a < b$ implies the existence of elements $c, d \in S$ such that $b = ca = ad$. A positively ordered semigroup S is called *archimedean* if $a^n < b$ for all positive integers n implies $a = e$ ($=$ the identity of S).[2] Finally, $a, b (\in S)$ are said to form an *anomalous pair* if $a \neq b, a^n < b^{n+1}$ and $b^n < a^{n+1}$ for all natural integers n.

Fully ordered semigroups have received some attention recently. The classical result of O. HÖLDER [4] which gives a sufficient condition that a f. o. semigroup S be embeddable in the additive semigroup P of all non-negative real numbers (with preservation of ordering) has been generalized in various ways. HÖLDER's conditions were:

(a) S is cancellative;

(b) S is naturally ordered;

(c) S is archimedean.

ALIMOV [1] replaced conditions (b) and (c) by the single one:

(d) S contains no anomalous pair,

thereby giving a necessary and sufficient condition for the embeddability of S in the real group. In § 1 we prove for positively ordered semigroups another necessary and sufficient condition in which the rather restrictive condition (a) is replaced by (c) and the rather weak condition (e) which relate more closely to the ordering relation:

(e) S contains no maximal element unless it consists of a single element.

[1] For the notions and basic facts related to f. o. semigroups we refer to CLIFFORD [3].

[2] We do not assume the existence of the identity e in S. If in a statement e occurs, then this means "the eventually existing identity".

A remarkable analogue of HÖLDER's theorem has been found by CLIFFORD [2]. He proved that if a f. o. semigroup S satisfies the conditions (b), (c) and the following two:

(a*) S is not cancellative;

(f) S is commutative,

then it is o-isomorphic (order-isomorphic) to a subsemigroup of one of the following two f. o. semigroups:

P_1: the real interval $[0, 1]$ with the operation: $a \circ b = \min(a+b, 1)$,

P_1^*: the interval $[0, 1]$ and the symbol ∞ with $a \circ b = a+b$ or ∞ according as $a+b \leq 1$ or >1.

In § 2 we shall show that conditions (b) and (c) imply (f), i. e. in CLIFFORD's theorem the hypothesis of commutativity can be omitted just as in HÖLDER's theorem. In the final § 3 we shall give a new proof of this theorem of CLIFFORD, one which seems to be more direct and simpler than CLIFFORD's original proof and which establishes at the same time HÖLDER's theorem too.

§ 1. Subsemigroups of the positive reals

This section is devoted to the proof of the following theorem:

THEOREM 1. *A necessary and sufficient condition that a positively f. o. semigroup be o-isomorphic to a subsemigroup of the additive semigroup of all non-negative real numbers is that it satisfy conditions* (c), (d) *and* (e).[3]

The necessity of these conditions being obvious, we may turn immediately to the proof of their sufficiency. Suppose therefore that (c), (d) and (e) are satisfied. We may assume that S contains more than one element.

Let $a, b \in S$ and $a \neq e$. We prove that $ab > b$ and $ba > b$. For, by (e), S contains an element c such that $c > b$, and because of (c) we can choose n so large that $a^n > c$. If $ab = b$ held, then also $a^n b = b$ whence $a^n b \geq a^n > c > b$ would be a contradiction. Thus $ab > b$, and similarly $ba > b$.

To prove commutativity, assume on the contrary that $ab \neq ba$. Then neither $a = e$ nor $b = e$, and what has been proved implies $(ab)^n < b(ab)^n a = (ba)^{n+1}$ and $(ba)^n < (ab)^{n+1}$, i. e. ab and ba form an anomalous pair, contrary to (d). Thus S is commutative.

Next assume that $ab = ac$ where $b < c$. By making use of commutativity, a simple induction shows that $ab^n = ac^n$ for all n. Surely, $c \neq e$, therefore $ac^n < ac^{n+1} = ab^{n+1}$, whence $c^n < b^{n+1}$ for all n. Since obviously $b^n \leq c^n < c^{n+1}$

[3] Mr. G. GRÄTZER noted that (c) and (e) may be united into the single condition: $a^n \leq b$ for all positive n implies $a = e$.

for all n, b and c form an anomalous pair. This contradiction proves that S is cancellative.

By ALIMOV's theorem [1], (a) and (d) imply what we wished to prove.

§ 2. The commutativity statement

We shall need the following

LEMMA. (CLIFFORD.) *Let S be a f. o. semigroup satisfying* (a*), (b) *and* (c). *Then*

1. *S contains a maximal element u,*
2. *for every $a \neq e$ there exists a natural integer k with $a^k = u$,*
3. *$ab = ac \neq u$ (or $ba = ca \neq u$) implies $b = c$.*

By hypothesis, three elements $a, b, c \in S$ exist such that $ab = ac$ and $b < c$ (or $ba = ca$ and $b < c$). By (b), $c = bx$ for some $x \in S$, $x \neq e$, therefore $y = yx$ holds for $y = ab$. If there existed an element $z > y$, $z \in S$, then choosing n so as to satisfy $x^n \geqq z$, we should have $y = yx^n \geq yz \geq z > y$, a contradiction. This establishes 1. and 3. at once. By (c), $a \neq e$ implies $a^k \geqq u$ for some k, and so $a^k = u$, completing the proof.

Now we are ready to prove:

THEOREM 2. *An archimedean, naturally f. o. semigroup is commutative.*

For the sake of simplicity we omit the identity from S if S has one; it is evident that this does not affect generality, because it must be the least element of S.

First assume that S possesses a minimal element a. Then to any $b \in S$, different from the eventually existing maximal element u, there exists an integer $k \geqq 1$ satisfying $a^k \leqq b < a^{k+1}$. Supposing $a^k < b$, there exists a $c \in S$ such that $b = a^k c$. But by the choice of $a, c \geqq a$, thus $b \geq a^k a > b$, which is absurd. Thus $b = a^k$ and S is a cyclic semigroup (generated by a).

Secondly assume that S has no minimal element. Then to any $x \in S$ there exists a $z \in S$ such that $z^2 \leqq x$; indeed, if $y < x$ and $x = yy_0$, then $z = \min(y, y_0)$ is a desired element. By way of contradiction, suppose that $ab > ba$ for some a, b in S. At first let[4] $ab < u$; then also $a < u, b < u$. If $ab = bax$ and $z^2 \leqq x$, then by (c) we can determine integers m, n satisfying $z^m \leqq a < z^{m+1}$ and $z^n \leqq b < z^{n+1}$. But these lead to the inequalities $ab = bax \geqq \geqq z^{n+m+2} > ab$ (in the non-cancellative case strict inequality because of the Lemma), a contradiction. Next let $ab = u$, and say $a < b$. Then $b < u$ (otherwise $ba = u = ab$) and $a^k < b < a^{k+1}$ (equality would imply that a and b

[4] If no maximal element u exists in S, then we may think of $ab < u$ to hold generally.

commute), for some $k \geqq 1$. Therefore $b = a^k c$ for a certain $c < a$, and since $ac \leqq b$ and $ca \leqq ba < ab = u$, we can apply what has been proved to conclude that a and c commute. This is again a contradiction, for then a and b also commute. Consequently, S must be commutative.

§ 3. Archimedean, naturally fully ordered semigroups

The next result is a generalization of HÖLDER's theorem, containing also CLIFFORD's theorem freed of the commutativity hypothesis.

THEOREM 3.[5] *Let S be an archimedean, naturally f. o. semigroup. Then S is o-isomorphic to a subsemigroup of P, P_1 or P_1^*.*

From the preceding theorem we know that S is necessarily commutative. If S is an infinite cyclic semigroup, generated by a, then $a^k \to k$ is an o-isomorphism of S into P. If S is a finite cyclic semigroup with the elements $(e<)a < a^2 < \cdots < a^n = a^{n+1}$, then the mapping $a^k \to \dfrac{k}{n}$ embeds S in P_1.

If S is not cyclic, then we again omit the eventually existing identity of S. By the proof of Theorem 2, to any $x \in S$ we can find a $z \in S$ satisfying $z^2 \leqq x$, and hence also one satisfying $z^t \leqq x$ for any preassigned integer $t > 0$. Now choose and fix an arbitrary $a \in S$ with $a < u$, and put $f(a) = 1$. To any $b \in S, b \neq u$, we define two sets of rational numbers: let L consist of all fractions m/n with $a \leqq x^n$ and $x^m \leqq b$ for some $x \in S$, and let k/l belong to the set U if $b \leqq y^k$ and $y^l \leqq a$ for some $y \in S$. The archimedean character of S guarantees that neither of L and U is empty. We show that $m/n \leqq k/l$. To any large t we can find a $z \in S$ with $z^t \leqq \min(x, y)$, hence $r \geqq t$ and $s \geqq t$ hold for the integers r, s defined by $z^r \leqq x < z^{r+1}$, $z^s \leqq y < z^{s+1}$. Therefore $z^{rm} \leqq b < z^{(s+1)k}$ and $z^{sl} \leqq a < z^{(r+1)n}$ whence $rm < (s+1)k$ and $sl < (r+1)n$. We infer that $\dfrac{m}{n} < \left(1 + \dfrac{1}{r}\right)\left(1 + \dfrac{1}{s}\right)\dfrac{k}{l}$ for arbitrarily large r, s, thus $m/n \leqq k/l$, in fact. On the other hand, the same argument shows that to any large $t > 0$ there exist $r, s \geqq t$ such that $s/(r+1) \in L$ and $(s+1)/r \in U$. Since the difference of these fractions tends to 0 with increasing t, it follows that there exists one and only one real number β such that $p \leqq \beta \leqq q$ for all $p \in L$ and $q \in U$. We put $f(b) = \beta$.

There is no difficulty in proving that the function f from $S \setminus u$ to the real axis is monotone and satisfies $f(b) = f(c)$ only if $b = c$. Moreover, we

[5] The proof of this theorem differs from the usual proofs of HÖLDER's theorem in that we must argue with 'small' elements rather than 'large' elements, due to the singular behaviour of the maximal element u.

have $f(bc)=f(b)+f(c)$ whenever $bc<u$ — which can again be proved by using sufficiently small elements $z\in S$.

If S is cancellative, then it contains no maximal element, and f is an o-isomorphism of S into P.

If S is not cancellative, then — in view of the Lemma — it contains a maximal element u. The set of values of $f(b)$ for all $b\in S\setminus u$ is bounded: if $a^i=u$, then i is an upper bound. Thus there exists a smallest real number α such that $f(b)\leqq\alpha$ for all $b\in S\setminus u$. If no $c\in S\setminus u$ exists with $f(c)=\alpha$, then set $f(u)=\alpha$, and if such a c exists, then let $f(u)=\infty$. The function $g(b)=\frac{1}{\alpha}f(b)$ is obviously an o-isomorphism of S into P_1 or P_1^*. This completes the proof.

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ON THE STRENGTH OF CONNECTEDNESS OF A RANDOM GRAPH

By

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Let G be a non-oriented graph without parallel edges and without slings, with vertices $V_1, V_2, \ldots, V_n$. Let us denote by $d(V_k)$ the *valency* (or degree) of a point V_k in G, i. e. the number of edges starting from V_k. Let us put

$$c(G) = \min_{1 \leq k \leq n} d(V_k). \tag{1}$$

If G is an arbitrary non-complete graph, let $c_p(G)$ denote the least number k such that by deleting k appropriately chosen vertices from G (i. e. deleting the k points in question and all edges starting from these points) the resulting graph is not connected. If G is a complete graph of order n, we put $c_p(G) = n-1$. Let $c_e(G)$ denote the least number l such that by deleting l appropriately chosen edges from G the resulting graph is not connected. We may measure the strength of connectedness of G by any of the numbers $c_p(G)$, $c_e(G)$ and in a certain sense (if G is known to be connected) also by $c(G)$. Evidently one has

$$c(G) \geqq c_e(G) \geqq c_p(G). \tag{2}$$

It is known further that any two points of G are connected by at least $c_p(G)$ paths having no point in common, except the two endpoints (theorem of MENGER—WHITNEY, see [1] and [2]) and by at least $c_e(G)$ paths having no edge in common (theorem of FORD and FULKERSON, see [3]).

We shall denote by $\nu_r(G)$ the number of vertices of G which have the valency r $(r = 0, 1, 2, \ldots)$.

As in two previous papers ([4], [5]) we consider the random graph $\Gamma_{n,N}$ defined as follows: Let there be given n labelled points $V_1, V_2, \ldots, V_n$. Let us choose at random N edges among the $\binom{n}{2}$ possible edges connecting these n points, so that each of the $\binom{\binom{n}{2}}{N}$ possible choices of these edges should be equiprobable. We denote by $\Gamma_{n,N}$ the random graph thus obtained. We shall denote by $\mathbf{P}(\cdot)$ the probability of the event in the brackets.

The aim of this note is to investigate the strength of connectedness of the random graph $\Gamma_{n,N}$ when n and N both tend to $+\infty$, $N=N(n)$ being a function of n. As it has been shown in [4], the following theorem holds:

THEOREM 1. *If we have* $N(n)=\frac{1}{2}n\log n+\alpha n+o(n)$ *where* α *is a real constant, then the probability of* $\Gamma_{n,N(n)}$ *being connected tends to* $\exp(-e^{-2\alpha})$ *for* $n\to+\infty$.

In this paper we shall prove the following theorem:

THEOREM 2. *If we have* $N(n)=\frac{1}{2}n\log n+\frac{r}{2}n\log\log n+\alpha n+o(n)$ *where* α *is a real constant and* r *a non-negative integer, then*

$$\lim_{n\to+\infty}\mathbf{P}(c_p(\Gamma_{n,N(n)})=r)=1-\exp\left(-\frac{e^{-2\alpha}}{r!}\right), \tag{3}$$

further

$$\lim_{n\to+\infty}\mathbf{P}(c_e(\Gamma_{n,N(n)})=r)=1-\exp\left(-\frac{e^{-2\alpha}}{r!}\right) \tag{4}$$

and

$$\lim_{n\to+\infty}\mathbf{P}(c(\Gamma_{n,N(n)})=r)=1-\exp\left(-\frac{e^{-2\alpha}}{r!}\right). \tag{5}$$

REMARK. Clearly Theorem 2 can be considered as a generalization of Theorem 1. As a matter of fact, any of the statements $c_p(G)=0$ or $c_e(G)=0$ is equivalent to G not being connected and thus for $r=0$ (3) and (4) reduce to the statement of Theorem 1. It has been shown further in [4] that if $N(n)=\frac{n}{2}\log n+\alpha n+o(n)$ and $\Gamma_{n,N(n)}$ is not connected, then it consists almost surely of a connected component and of a few isolated points. Therefore (5) is for $r=0$ also equivalent to the statement of Theorem 1. Thus in proving Theorem 2 we may restrict ourselves to the case $r\geqq 1$.

The statement (5) of Theorem 2 gives information about the *minimal* valency of points of $\Gamma_{n,N}$. In a forthcoming note we shall deal with the same question for larger ranges of N (when $c(\Gamma_{n,N})$ tends to infinity with n), further with the related question about the *maximal* valency of points of $\Gamma_{n,N}$.

We shall prove further the following

THEOREM 3. *If we have* $N(n)=\frac{1}{2}n\log n+\frac{r}{2}n\log\log n+\alpha n+o(n)$ *where* α *is a real constant and* r *a non-negative integer, then we have*

$$\lim_{n\to+\infty}\mathbf{P}(\nu_r(\Gamma_{n,N(n)})=k)=\frac{\lambda^k e^{-\lambda}}{k!}\quad \textit{for}\quad k=0,1,\dots \tag{6}$$

where $\lambda = \frac{e^{-2\alpha}}{r!}$; *in other words, the distribution of* $\nu_r(\Gamma_{n,N(n)})$ *tends to a Poisson distribution.*

PROOF OF THEOREMS 2 AND 3. Let $r \geqq 1$ be an integer and $-\infty < \alpha < +\infty$. Let us suppose that

$$N(n) = \frac{1}{2} n \log n + \frac{r}{2} n \log\log n + \alpha n + o(n). \tag{7}$$

Let $\Gamma_{n,N}$ be a random graph with the n vertices $V_1, V_2, \ldots, V_n$ and having N edges. Let $P_k(n, N, r)$ denote the probability that by removing r suitably chosen points from $\Gamma_{n,N}$ there remain two disjoint graphs, consisting of k and $n-k-r$ points, respectively. We may suppose $k < \left[\frac{n-r}{2}\right]$. First we have clearly

$$P_k(n, N, r) \leqq \binom{n}{r}\binom{n-r}{k} \frac{\binom{\binom{n}{2} - k(n-k-r)}{N}}{\binom{\binom{n}{2}}{N}}.$$

It follows by some obvious estimations that

$$\sum_{(r+3)\frac{\log n}{\log\log n} < k \leqq \left[\frac{n-r}{2}\right]} P_k(n, N(n), r) = O\left(\frac{1}{n}\right). \tag{8}$$

Now we consider the case $k \leqq (r+3)\frac{\log n}{\log\log n}$. Let $P_k^*(n, N, r)$ denote the probability that by removing r suitably chosen points (the set of which will be denoted by Δ) $\Gamma_{n,N}$ can be split into two disjoint subgraphs Γ' and Γ'' consisting of k and $n-k-r$ points, respectively, but that $\Gamma_{n,N}$ can not be made disconnected by removing only $r-1$ points. If $\Gamma_{n,N}$ has these properties and if s denotes the number of edges of $\Gamma_{n,N}$ connecting a point of Δ with a point of Γ', then we have clearly $s \geqq r$. Otherwise, by definition, $s \leqq rk$. Thus we have

$$P_k^*(n, N, r) \leqq \sum_{s=r}^{rk} \binom{n}{r}\binom{n-r}{k}\binom{rk}{s} \frac{\binom{\binom{n}{2} - k(n-k)}{N-s}}{\binom{\binom{n}{2}}{N}}. \tag{9}$$

It follows that

$$\sum_{k=2}^{\left[(r+3)\frac{\log n}{\log\log n}\right]} P_k^*(n, N(n), r) = O\left(\frac{1}{\log n}\right). \tag{10}$$

From (8) and (10) it follows that for $n \to +\infty$

$$\mathbf{P}(c_p(\Gamma_{n, N(n)}) = r) \sim \mathbf{P}(c(\Gamma_{n, N(n)}) = r). \tag{11}$$

As a matter of fact, (8) and (10) imply that if by removing r suitably chosen points (but not by removing less than r points) $\Gamma_{n, N(n)}$ can be split into two disjoint subgraphs Γ' and Γ'' consisting of k and $n-k-r$ points, respectively, where $k \leqq \left[\frac{n-r}{2}\right]$, then only the case $k=1$ has to be considered, the probability of $k>1$ being negligibly small. It remains to prove (5). This can be done as follows. First we prove that

$$\lim_{n \to +\infty} \mathbf{P}(c(\Gamma_{n, N(n)}) \leqq r-1) = 0. \tag{12}$$

For $r=1$ this follows already from Theorem 1. Thus we may suppose here $r \geqq 2$. We have

$$\mathbf{P}(c(\Gamma_{n, N}) \leqq r-1) \leqq \sum_{h=1}^{r-1} n \binom{n-1}{h} \frac{\binom{\binom{n}{2}-(n-1)}{N-h}}{\binom{\binom{n}{2}}{N}},$$

and thus

$$\mathbf{P}(c(\Gamma_{n, N(n)}) \leqq r-1) = O\left(\frac{1}{\log n}\right) \tag{13}$$

which proves (12).

Now let $\nu_r(\Gamma_{n, N})$ denote the number of vertices of $\Gamma_{n, N}$ which have the valency r. Then we have clearly by (12)

$$\mathbf{P}(c(\Gamma_{n, N(n)}) = r) \sim \mathbf{P}(\nu_r(\Gamma_{n, N(n)}) \neq 0). \tag{14}$$

Now evidently

$$\mathbf{P}(\nu_r(\Gamma_{n, N(n)}) \neq 0) = \sum_{j=1}^{n} (-1)^{j-1} S_j \tag{15}$$

where

$$S_j = \sum\sum_{1 \leqq k_1 < k_2 < \cdots < k_j \leqq n} \cdots \sum \mathbf{P}(d(V_{k_1}) = r, d(V_{k_2}) = r, \ldots, d(V_{k_j}) = r). \tag{16}$$

Evidently, if we stop after taking an even or odd number of terms of the

sum on the right-hand side of (15), we obtain a quantity which is greater or smaller, respectively, than the left-hand side of (15). Now clearly

$$\mathbf{P}(d(V_k)=r)=\binom{n-1}{r}\frac{\binom{\binom{n}{2}-(n-1)}{N(n)-r}}{\binom{\binom{n}{2}}{N(n)}}\sim\frac{e^{-2\alpha}}{nr!},$$

and thus

$$\lim_{n\to+\infty} S_1=\frac{e^{-2\alpha}}{r!}. \tag{17}$$

Now let us consider $\mathbf{P}(d(V_{k_1})=r,\, d(V_{k_2})=r)$ where $k_1\neq k_2$. If both V_{k_1} and V_{k_2} have valency r, three cases have to be considered: a) either V_{k_1} and V_{k_2} are not connected, and there is no point which is connected with both V_{k_1} and V_{k_2}; b) or V_{k_1} and V_{k_2} are not connected, but there is a point connected with both; c) V_{k_1} and V_{k_2} are connected. We denote the probabilities of the corresponding subcases by $\mathbf{P}_a(d(V_{k_1})=r,\, d(V_{k_2})=r)$, $\mathbf{P}_b(d(V_{k_1})=r,\, d(V_{k_2})=r)$ and $\mathbf{P}_c(d(V_{k_1})=r,\, d(V_{k_2})=r)$, respectively. We evidently have

$$\mathbf{P}_a(d(V_{k_1})=r,\, d(V_{k_2})=r)=\frac{(n-2)!}{r!^2(n-2r-2)!}\frac{\binom{\binom{n}{2}-(2n-3)}{N(n)-2r}}{\binom{\binom{n}{2}}{N(n)}}\sim\left(\frac{e^{-2\alpha}}{n\cdot r!}\right)^2,$$

and thus

$$\sum\sum_{1\le k_1<k_2\le n}\mathbf{P}_a(d(V_{k_1})=r,\, d(V_{k_2})=r)\sim\frac{1}{2}\left(\frac{e^{-2\alpha}}{r!}\right)^2. \tag{18}$$

On the other hand (denoting by l the number of points which are connected with both V_{k_1} and V_{k_2}), we have

$$\mathbf{P}_b(d(V_{k_1})=r,\, d(V_{k_2})=r)=$$

$$=\sum_{l=1}^{r}\frac{(n-2)!}{l!(r-l)!(n-2r+l-2)!}\frac{\binom{\binom{n}{2}-(2n-3)}{N(n)-2r}}{\binom{\binom{n}{2}}{N(n)}}=O\left(\frac{1}{n^3}\right). \tag{19}$$

Similarly one has

$$\mathbf{P}_c(d(V_{k_1})=r,\, d(V_{k_2})=r)=$$

(20)
$$=\sum_{l=0}^{r-1}\frac{(n-2)!}{l!(r-l-1)!^2(n-2r+l)!}\frac{\binom{\binom{n}{2}-(2n-3)}{N(n)-2r}}{\binom{\binom{n}{2}}{N(n)}}=O\left(\frac{1}{n^4}\right).$$

Thus we obtain

$$\lim_{n\to+\infty} S_2=\frac{1}{2}\left(\frac{e^{-2\alpha}}{r!}\right)^2.$$

The cases $j>2$ can be dealt with similarly. Thus we obtain

(21)
$$\lim_{n\to+\infty} S_j=\frac{1}{j!}\left(\frac{e^{-2\alpha}}{r!}\right)^j \qquad (j=1,2,3,4,\ldots).$$

It follows from (16) and (21) that

(22)
$$\lim_{n\to+\infty}\mathbf{P}(\nu_r(\Gamma_{n,N(n)})\neq 0)=1-\exp\left(-\frac{e^{-2\alpha}}{r!}\right).$$

In view of (2), (11) and (14) Theorem 2 follows.

To prove Theorem 3 it is sufficient to remark that by the well-known formula of CH. JORDAN

(23)
$$\mathbf{P}(\nu_r(\Gamma_{n,N(n)})=k)=\sum_{j=0}^{n-k}(-1)^j\binom{j+k}{j}S_{j+k},$$

and thus by (21), putting $\lambda=\dfrac{e^{-2\alpha}}{r!}$, we obtain for $k=0,1,\ldots$

(24)
$$\lim_{n\to+\infty}\mathbf{P}(\nu_r(\Gamma_{n,N(n)})=k)=\frac{\lambda^k}{k!}\sum_{j=0}^{\infty}\frac{(-1)^j\lambda^j}{j!}=\frac{\lambda^k e^{-\lambda}}{k!}.$$

Thus Theorem 3 is proved.

Our thanks are due to T. GALLAI for his valuable remarks.

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